18.2. SUBSTITUTION THEOREM. If G and H are recursive, there is a recursive F such that

$$(4) \qquad F(\vec{\alpha},\vec{x}) \simeq G(\lambda z H(z,\vec{\alpha},\vec{x}),\ \vec{\alpha},\ \vec{x})$$

for all $\vec{\alpha},\vec{x}$ such that $\lambda z H(z,\vec{\alpha},\vec{x})$ is a real. In particular, if H is total, then the F defined by (4) is recursive.

Proof. Let g be an index of G. If $\lambda z H(z,\vec{\alpha},\vec{x})$ is a real, then the right side of (4) is, by (1),

$$U(\mu y T_{m,k}(g,\vec{x},y,\overline{H}(y,\vec{\alpha},\vec{x}),\overline{\vec{\alpha}}(y))).$$

We can use this as our definition of $F(\vec{\alpha},\vec{x})$. □

In particular, it follows that λy is a recursive expression when it is used in front of an expression defined for all values of y.

REMARK. If H is not total, there may be $\vec{\alpha},\vec{x}$ such that $F(\vec{\alpha},\vec{x})$ is defined, but such that $\lambda z H(y,\vec{\alpha},\vec{x})$ is not a real and hence such that the right side of (4) is not defined.

The results of §13 and §14 extend without difficulty. However, in §13 it is natural to consider a further extension in which we allow quantifiers on real variables. We investigate this in the next section.

19. The Analytical Hierarchy

A relation is <u>analytical</u> if it has an explicit definition with a prefix consisting of quantifiers, which may be either universal or existential and may be on either number variables or real variables, and a recursive matrix. The basic theory of analytical relations is due to Kleene.

We begin with some rules for simplifying prefixes. As before, these may change the matrix, but they leave it recursive.

Two quantifiers are of the <u>same kind</u> if they are both universal or both existential; they are of the <u>same type</u> if they are both on real variables or both on