17. Large RE Sets

Post's idea for solving his problem was to show that a sufficiently large RE set could not have degree **0'**, and then producing a non–recursive RE set which was this large. Although this idea did not solve Post's problem, it led to many interesting results, which we shall explore briefly.

We introduce our first type of large RE set. A simple set is an RE set whose complement is infinite but includes no infinite RE set. Note that a simple set is not recursive, since its complement is not RE. We shall see later that every RE degree other than **0** is the degree of a simple set; so we cannot show that a simple set cannot have degree **0'**. However, we shall prove the weaker result that a simple set cannot be RE complete. In order to do this, we first find a characterization of the RE complete sets.

A creating function for a set A is a recursive real F such that $F(e) \in A \leftrightarrow F(e) \in W_e$ for all e. A creative set is an RE set which has a creating function. For example, if A is defined by $e \in A \leftrightarrow e \in W_e$, the A is creative with creating function I_1^1. A creative set is in some sense effectively non–recursive; for the creating function F shows that A^c is not equal to any W_e and hence is not RE.

17.1. PROPOSITION. A set is RE complete iff it is creative.

Proof. Let A be RE complete. Then there is a recursive real F such that $W_e(e) \leftrightarrow F(e) \in A$. Using the RE Parameter Theorem, pick a recursive total S so that $W_{S(e)}(x) \leftrightarrow W_e(F(x))$. Then

$$F(S(e)) \in A \leftrightarrow W_{S(e)}(S(e)) \leftrightarrow F(S(e)) \in W_e.$$

Hence $G(e) = F(S(e))$ defines a creating function for A.

Suppose that A is creative with creating function F, and let B be any RE set. Pick a total recursive S so that $W_{S(x)}(y) \leftrightarrow x \in B$. Then

$$x \in B \leftrightarrow F(S(x)) \in W_{S(x)} \leftrightarrow F(S(x)) \in A;$$

so B is reducible to A. □

17.2. PROPOSITION. A simple set is not RE complete.