definition of this function has a new clause for each r:

$$Reg(j,e,x,n+1) = H_r(Reg((i)_1,e,x,n)) \text{ if } (i)_0 = 3 \ \& \ (i)_3 = r \ \& \ (i)_2 = j.$$

This means that in the definition of $T_k^{\Phi}(e,\vec{x},y)$, H_r appears only in contexts $H_r(X)$ where X designates a number appearing in a register during the P–computation from $\vec{x}$ and hence $< y$. Thus we may replace $H_r(X)$ by $(\overline{H_r}(y))_X$.

If Φ is $H_1,\dots,H_m$, we write $\overline{\Phi}(z)$ for $\overline{H_1}(z),\dots,\overline{H_m}(z)$. The above can be summarized as follows: there is a recursive relation $T_{k,m}$ such that

$$(1) \qquad T_k^{\Phi}(e,\vec{x},y) \leftrightarrow T_{k,m}(e,\vec{x},y,\overline{\Phi}(y)).$$

Thus if $\{e\}^{\Phi}(\vec{x}) \simeq z$ with computation number y, and $\overline{\Phi}(y) = \overline{\Phi'}(y)$, then $\{e\}^{\Phi'}(\vec{x}) \simeq z$.

13. The Arithmetical Hierarchy

We are now going to study the effect of using unbounded quantifiers in definitions of relations. From now on, we agree that n designates a non–zero number. The results of this section are due to Kleene.

A relation R is <u>arithmetical</u> if it has an explicit definition

$$(1) \qquad R(\vec{x}) \leftrightarrow Qy_1 \dots Qy_n P(\vec{x},y_1,\dots,y_n)$$

where each Qy_i is either $\exists y_i$ or $\forall y_i$ and P is recursive. We call $Qy_1 \dots Qy_n$ the <u>prefix</u> and $P(\vec{x},y_1,\dots,y_n)$ the <u>matrix</u> of the definition. We are chiefly interested in the prefix, since it measures how far the definition is from being recursive.

We shall first see how prefixes can be simplified. As z runs through all number, $(z)_0,(z)_1$ runs through all pairs of numbers. It follows that

$$\forall x \forall y R(x,y) \leftrightarrow \forall z R((z)_0,(z)_1)$$

and

$$\exists x \exists y R(x,y) \leftrightarrow \exists z R((z)_0,(z)_1).$$

Using these equivalences, we can replace two adjacent universal quantifiers in a prefix by a single such quantifier, and similarly for existential quantifiers. For example, a definition