An Ω–relation is an expression $X = Y$ where X and Y are Ω–words. Then if S is an Ω–semigroup, $X = Y$ is either true or false in S. Now let R be a finite set of Ω–relations and let K be an Ω–relation. Then $R \Rightarrow K$ means that K is true in every Ω–semigroup in which all of the relations in R are true. The word problem for Ω–semigroups is to find an algorithm by which, given R and K, we can decide if $R \Rightarrow K$.

We shall show that the word problem for Ω–semigroups is unsolvable. (This was proved independently by Post and Markov.) Let W' be the symmetric process constructed above. Let R consist of the relations $X = Y$ such that $X \rightarrow Y$ is in W' (and hence $Y \rightarrow X$ is in W'). We shall show that $X \Rightarrow_{W'} Y$ iff $R \Rightarrow X = Y$. Hence the word problem for Ω–semigroups is unsolvable even for this particular R.

Clearly $X \Rightarrow_{W'} Y$ implies $R \Rightarrow X = Y$. To prove the implication in the other direction, we construct an Ω–semigroup. First note that the relation $X \Rightarrow_{W'} Y$ between X and Y is an equivalence relation on the class of Ω–words; this follows from the fact that W' is symmetric. Let X^* be the equivalence class of X. Let S be the set of all these equivalence classes; and define a binary operation $\cdot$ on S by $X^* \cdot Y^* = (XY)^*$ (where XY is X followed by Y). A little thought shows that $(XY)^*$ depends only on the equivalence classes X^* and Y^*; so our definition makes sense. It is easy to see that S is then a semigroup; the unit element is the equivalence class of the empty word.

We make S into an Ω–semigroup by letting the symbol a represent a^*; the word X then represents X^*. If $X = Y$ is in R, then X and Y are equivalent; so $X^* = Y^*$; so $X = Y$ is true in S. It follows that if $R \Rightarrow X = Y$, then $X = Y$ is true in S and hence $X \Rightarrow_{W'} Y$. This completes our proof.

11. Undecidable Theories

We shall see how some problems of the following type can be shown to be