ON SIMILARITIES OF COMPLETE THEORIES

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In classical model theory two objects of different nature correspond to every signature σ:

L — the first-order language of σ and
K — the class of all structures of σ.

But there is a well-known one-to-one correspondence between maximal consistent sets of sentences of L and minimal axiomatizable classes in L of structures from K. When we say that we study the complete theory T we usually mean the pair $\langle T, \mathrm{Mod}(T)\rangle$, where $\mathrm{Mod}(T)$ is the class of all models of T. In connection with this duality of the nature of complete theories I want to introduce two notions of similarity which play the role of isomorphisms and two notions of nearness of theories.

§1. Syntactical similarity.

Let $F_n(T)$, $n < \omega$, be the Boolean algebras of formulas of T with exactly n free variables $v_1, \dots, v_n$, and $F(T) = \bigcup_n F_n(T)$.

DEFINITION 1. Complete theories T_1 and T_2 are *syntactically similar* if and only if there exists a bijection $f : F(T_1) \to F(T_2)$ such that

(i) $f \restriction F_n(T_1)$ is an isomorphism of the Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;
(ii) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;
(iii) $f(v_1 = v_2) = (v_1 = v_2)$.

EXAMPLE 1. The following theories T_1 and T_2 of the signature $\sigma = \langle \varphi, \psi \rangle$ are syntactically similar, where φ, ψ are binary functions:

$$T_1 = \mathrm{Th}(\langle \mathbf{Z}; +, \cdot \rangle), \qquad T_2 = \mathrm{Th}(\langle \mathbf{Z}; \cdot, + \rangle).$$

§2. Semantic similarity.

From the point of view of a model-theoretician, the object $\langle \mathrm{Mod}(T); \simeq, \preccurlyeq \rangle$ is important for the study of the class $\mathrm{Mod}(T)$. Properties of this object are more completely characterized by the triple $\langle \mathfrak{C}, \mathrm{Aut}(\mathfrak{C}), \mathcal{N}(\mathfrak{C})\rangle$, where $\mathfrak{C}$ is the monster-model of T, $\mathrm{Aut}(\mathfrak{C})$ is the group of all automorphisms of $\mathfrak{C}$ and $\mathcal{N}(\mathfrak{C})$ is the class of all elementary substructures of $\mathfrak{C}$. Therefore the following definition of semantic similarity is justified.

I shall begin with some preliminary notions.