A TRANSFINITE VERSION OF PUISEUX'S THEOREM, WITH APPLICATIONS TO REAL CLOSED FIELDS

M. H. MOURGUES and J.-P. RESSAYRE

Abstract. Extending the effective version of Puiseux's theorem, we compute the roots of polynomials inside all fields of the form $k((x))^\Gamma$, where k is real closed and Γ divisible. We use this computation to prove that every real closed field has an integer part, that is, a discrete subring which plays for the field the same role as $\mathbf{Z}$ plays for $\mathbf{R}$.

Puiseux's theorem in its detailed form allows one to compute the roots of a polynomial, inside the field of "Puiseux series"; Lemma 3.6 and Remark 3.7 below generalize this computation to the case of a field $k((x))^\Gamma$, where k is a real closed field and Γ a divisible ordered abelian group, even if Γ is non-archimedian.

Two applications are given: the truncation lemma of F. Delon (see 3.5 below), and the existence of an "integer part" in every real closed field (see 1.4 below). Another proof of these results appears in [MR], but without the generalization of Puiseux's theorem which is one of the chief interests of this paper. The first author has developed extensions and other applications of such computations in [M].

§1. Definitions, remarks.

1.1. Let us denote by $\widetilde{K}$ the real closure of a totally ordered field K.

1.2. We say that a subring Z of a ring A is an *integer part* of A if it is discrete and if for any $x \in A$, there is $z \in Z$ such that $z \leq x < z + 1$. We call this unique element z the *integer part* of x and write $z = [x]$.

1.3. S. Boughattas showed in [B] that on the one hand, every totally ordered field has an ultrapower endowed with an integer part, and on the other hand, there are ordered fields without integer parts. In fact, he has for every integer p, a p-real closed field with no integer part. We show that these examples are optimal, in the sense that every real closed field has an integer part.

1.4. In fact, we will prove a stronger result:

Let A be a convex subfield of $K = \widetilde{K}$. Then any integer part Z of A can be extended to an integer part Z_K of K.