NEW FOUNDATIONS FOR MATHEMATICAL THEORIES

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§1. **The motivation.** In this paper, I shall outline a new approach to the logical foundations of mathematical theories. One way of looking at its motivation is as follows (I am following here Hintikka, 1989):

In the foundational work around 1900, e.g. in Hilbert's ***Foundations of geometry***, a crucial role was played by assumptions of *extremality* (i.e., *minimality* and *maximality*). For instance, Hilbert's so-called axiom of completeness is a maximality assumption. The Archimedean axiom can be thought of as a minimality assumption, the principle of induction likewise as a minimality axiom, and Dedekind's assumption of the existence of a real for each cut as a maximality assumption. Slowly, it has become clear to everybody that such extremality assumptions cannot normally be expressed as ordinary first-order axioms. To what extent they can or cannot be expressed in other ways, e.g. as higher-order axioms or set-theoretical axioms, and to what extent we should try to express them in such ways, will not be discussed here. In any case, in spite of the tremendous *prima facie* interest and power of extremality assumptions, they have not attracted much interest lately.

The approach proposed and outlined here relies crucially on extremality assumptions but seeks to implement them in a new way on a first-order level. Instead of introducing extremality assumptions on the top of a ready-made logic as explicit axioms, I propose to build them into the very logic we are employing, thus by-passing the difficulties the earlier uses of extremality assumptions encountered.

A logic is in effect specified by a space Ω of models together with a definition of what it means for a statement (closed formula) to be true in a model $M \in \Omega$ (and for a formula to be satisfied with in M). I shall not modify the latter ingredient. Instead, I propose to modify the usual space of models (of a given first-order language L) in the simplest possible way, viz. by omitting some of its members.

Even though this kind of modification looks innocuous, it facilitates a radical new look at the prospects of mathematical and logical theories. Most importantly, the possibility of reaching completeness can be profoundly affected.

What are the different kinds of completeness relevant here? Here are four candidates, which have not always been distinguished from each other sufficiently clearly:

(1) *Descriptive completeness.* It is an attribute of a *non-logical theory.* It means that the theory has as its models only the intended (standard) ones, i.e., that it has no non-standard ones. If there is only one standard