DEFINABILITY AND GLOBAL DEGREE THEORY[1]

S. BARRY COOPER

Gödel's work [Gö34] on undecidable theories and the subsequent formalisations of the notion of a recursive function ([Tu36], [Kl36] etc.) have led to an ever deepening understanding of the nature of the non-computable universe (which as Gödel himself showed, includes sets and functions of everyday significance). The nontrivial aspect of Church's Thesis (any function not contained within one of the equivalent definitions of recursive/Turing computable, cannot be considered to be effectively computable) still provides a basis not only for classical and generalised recursion theory, but also for contemporary theoretical computer science. Recent years, in parallel with the massive increase in interest in the computable universe and the development of much subtler concepts of 'practically computable,' have seen remarkable progress with some of the most basic and challenging questions concerning the non-computable universe, results both of philosophical significance and of potentially wider technical importance.

Relativising Church's Thesis, Kleene and Post [KP54] proposed the now standard framework of the *degrees of unsolvability* $\mathcal{D}$ as the appropriate fine structure theory for ω^ω. A technical basis was found in the various equivalent notions of relative computability provided by Turing [Tu39], Kleene [Kl43], Post [Po43] and others. Within the study of $\mathcal{D}$ it has become usual to distinguish (see [Sh81]) two approaches: that of *global degree theory*, based more or less on a number of general questions concerning the structure of the degrees first stated by Rogers in his book [Ro67]; and that of *local degree theory* with its emphasis on degree structure not far removed from the degree $\mathbf{0}$ of recursive functions (in particular the *recursively enumerable*—or *r.e.*—degrees and the degrees below $\mathbf{0}'$—the degree of the coded theorems of Peano arithmetic). Of course, there is an intimate relationship between the two approaches, and the aim here is to describe some recent results showing how even the most archetypal local degree theory can be used to resolve interesting and important global questions.

§1. Notation and terminology.

We use standard notation and terminology (see for example [So87]).

For instance, corresponding to the ith Turing machine, Φ_i denotes the ith partial recursive (p.r.) functional $2^\omega \to 2^\omega$. A set A is *Turing reducible to* a set B ($A \leq_T B$) if and only if $A = \Phi_i^B$ for some $i \in \omega$, and A, B are *Turing equivalent*

[1]Research partially supported March–June 1989 by a C.N.R. visiting professorship at the University of Siena. We are grateful to Andrea Sorbi and Bob di Paola for encouragement during the early stages of this work.