ON THE GEOMETRY OF U-RANK 2 TYPES

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Abstract. Let T be a countable superstable theory with $< 2^{\aleph_0}$ countable models. We solve the algebraic problem from [Ne4, §4]. In particular, in some cases we complete the countable classification of skeletal p of U-rank 2 (cf. [Bu4]).

§0. Introduction. Throughout the paper we assume that T is a complete countable superstable theory with $< 2^{\aleph_0}$ countable models. For the background from stability theory see [Sh], [Ba], [Bu1], or [P]. The results in [Bu2] suggest that if T has infinite U-rank then every countable model M of T is determined by a subset A of M, called its skeleton (cf. [Bu4]). Hence in the course of proving Vaught's conjecture we have to determine possible isomorphism types of skeletons. The easiest non-trivial case we faced in [Bu4] and [Ne4] was as follows. Assume $p \in S(\emptyset)$ is stationary, non-isolated, has U-rank 2, and if b realizes p then for some $a \in \mathrm{acl}(b)$, $U(a) = 1$ and $\mathrm{tp}(b/a)$ is non-isolated. Let $I(p, \kappa)$ be the number of isomorphism types of sets $p(M)$ of power κ, where M is a model of T. We wanted to prove that $I(T, \aleph_0) < 2^{\aleph_0}$ implies $I(p, \aleph_0) \leq \aleph_0$. Anyway, considering $I(p, \aleph_0)$ seems to be a necessary step on a way to prove Vaught's conjecture for superstable T. Let us recall the main path of reasoning from [Bu4] and [Ne4] thus far.

For a, b as above let $q = \mathrm{tp}(a/\emptyset)$ and $p_a = \mathrm{tp}(b/a)$. We want to count, up to isomorphism of the monster model $\mathfrak{C}$, the number of sets $p(M)$, where M is countable. $p(M)$ is the union of sets $p_{a'}(M)$, where $a' \in M$ realizes q. q has finite multiplicity hence by adding an element of $\mathrm{acl}(\emptyset)$ to the signature we can assume that q is stationary. Throughout we assume $T = T^{\mathrm{eq}}$. Further on in determining the structure of $p(M)$ we can easily dispose with the cases when p_a is strongly minimal or trivial. Hence we can assume that p_a is properly minimal and non-trivial. Then [Ne1] implies that p_a has finite multiplicity, and [Bu1] gives that every stationarization of p_a is locally modular. Similarly we can assume that for b realizing p_a, $\mathrm{stp}(b/a)$ is not modular, non-orthogonal to $\emptyset$ and almost orthogonal to $\emptyset$. In particular, p_a is weakly orthogonal to $q \mid a$. Also, we can assume that all stationarizations of types p_a, $a \in q(\mathfrak{C})$, are non-orthogonal. If $q(M)$ has finite acl-dimension then $p(M)$ can be characterized up to isomorphism just as in [Bu2]. Hence we can assume that for every countable M we consider, $q(M)$ has dimension $\aleph_0$, and $Q = q(M)$ is fixed. As $p(M) = \bigcup\{p_a(M) : a \in Q\}$, classifying the structure of $p(M)$ amounts to describing how the weakly minimal sets $p_a(M)$, $a \in Q$, can be arranged together to form $p(M)$. The types p_a, $a \in Q$, are non-orthogonal, so the main difficulty lies in that we are not free in deciding