§9. Uniqueness of the Next Extender

In §11 we shall construct an extender sequence $\vec{E}$ such that $L[\vec{E}] \models$ "there is a Woodin cardinal, and every level $\mathcal{J}_\alpha^{\vec{E}}$ of $L[\vec{E}]$ is a 1-small coremouse". The sequence $\vec{E}$ will be defined by recursion. The recursion is substantially more subtle than it is for sequences of measures, but the basic idea is still to define $\vec{E}_\gamma$ by recursion on γ, by making $\vec{E}_\gamma$ be the least extender which can be added to the sequence $\vec{E} \restriction \gamma$ so that the extender sequence remains good. Part of the strategy will be to pick $\vec{E}_\gamma$ without regard to the initial segment condition and then prove that in fact it does satisfy the initial segment condition as well. We would like to show that there is always only one possible choice of $\vec{E}_\gamma$ for each γ, so that if ρ is the natural length of $\vec{E}_\gamma \restriction \rho$ and G of length γ' is its trivial extension then G, being a legal choice for $E_{\gamma'}$, must in fact be $E_{\gamma'}$. Of course this ignores the second alternative in the initial segment condition, but more important we are unable to prove this uniqueness: so far as we know there could be one choice of types I or III and a second of type II. In this section we will prove uniqueness for types I or III, and in section 11 this will be used for the case when ρ is a cardinal in $L[\vec{E}]$. In section 10 we will prove a related result which will apply in the cases when ρ is not a cardinal in $L[\vec{E}]$.

The standard method for showing uniqueness of the next extender on the sequence involves *Doddages* and comparison of a Doddage with itself. The method originates in Mitchell's [M74R], see also [D]. We need only a simple sort of Doddage, dubbed by Jensen a bicephalus. A bicephalus is like an active premouse, except that it has two predicates corresponding to two candidates for a last extender. By comparing bicephali with themselves we show that in sufficiently iterable bicephali, these candidates are not distinct.

Unfortunately, when we want to form an ultrapower of a bicephalus whose last extenders differ in type, we have a problem. We may want to squash for the sake of one extender, but if we do so it is not clear how to carry along the other. This is the reason we will also need the alternative technique from section 10.

The first problem in dealing with bicephali will be to verify that when we form the ultrapower of a bicephalus both of whose last extenders are of type III, the squashing procedures in the two cases are consistent with one another. We shall verify this now, in Lemma 9.1.

If M is an active ppm then $\nu^{\mathcal{M}}$ is just the the natural length of the extender coded by $\dot{F}^{\mathcal{M}}$, that is if $\mathcal{M}$ is of type II or III then ν^{M} is the strict sup of its generators, while if $\mathcal{M}$ is type I, then $\nu^{\mathcal{M}} = (\kappa^+)^{\mathcal{M}}$.)

Lemma 9.1. *Let $\mathcal{M}$ be a type III ppm, and G an extender over M with* $\operatorname{crit} G = \kappa < \nu^{\mathcal{M}}$. *Let $\mathcal{P}$ be the ultrapower of $\mathcal{M}$ via G, where functions in $|\mathcal{M}|$ are used, and let $i : \mathcal{M} \to \mathcal{P}$ be the canonical embedding. Assume $\mathcal{P}$ is wellfounded. Let $\nu^* = sup\ i''\nu^{\mathcal{M}}$. Then*