In this section we prove the central fine structural result of the theory we are developing, namely that every 1-small mouse is k-solid for all k. We also derive, by the same method, some condensation results we shall need later. Our proofs of these facts trace back to Dodd's proof that the models of [D] satisfy the GCH.

For mice $\mathcal{M}$ up to a strong cardinal (that is, for mice $\mathcal{M}$ such that $\mathcal{J}_\kappa^{\mathcal{M}} \models$ "There are no strong cardinals" whenever $\kappa = \operatorname{crit} E$ for some extender E on the $\mathcal{M}$ sequence), our proof actually shows that $\mathfrak{C}_k(\mathcal{M})$ is an iterate of $\mathfrak{C}_{k+1}(\mathcal{M})$, with the iteration map having critical point $\geq \rho_{k+1}(\mathcal{M})$. That is, every "very small" mouse is an iterate of its core. We suspect that this is not true for arbitrary 1-small mice.

Recall that $u_0(\mathcal{M}) = \varnothing$, and that $u_k(\mathcal{M}) = \langle \rho_k(\mathcal{M}), b_0, \cdots, b_S, \rho_{k-1}^{\mathcal{M}} \rangle$ for $k \geq 1$, where $b_0 \cdots b_S$ are the solidity witnesses for $p_k(\mathcal{M})$ and the last coordinate $\rho_{k-1}^{\mathcal{M}}$ occurs only if it is defined and is smaller than $\mathrm{OR}^{\mathcal{M}}$. Thus $p_{k+1}(\mathcal{M})$ is the appropriate collapse of $\langle r, u_k(\mathcal{M}) \rangle$, where r is the $k+1$st standard parameter of $(\mathfrak{C}_k(\mathcal{M}), u_k(\mathcal{M}))$.

Recall that if $\pi : \mathcal{M} \to \mathcal{N}$ is a k-embedding, then $\pi(u_k(\mathcal{M})) = u_k(\mathcal{N})$.

Theorem 8.1. *Let $\mathcal{M}$ be a k-sound, 1-small, k-iterable premouse, where $k < \omega$. Let r be the $k+1$st standard parameter of $(\mathcal{M}, u_k(\mathcal{M}))$. Then r is $k+1$-solid and $k+1$ universal over $(\mathcal{M}, u_k(\mathcal{M}))$.*

PROOF. Let $u = u_k(\mathcal{M})$ and $r = \langle \alpha_0, \cdots, \alpha_S \rangle$, with the ordinals α_s in decreasing order. Let $\alpha_{S+1} = \rho_{k+1}^{\mathcal{M}}$. Let $s \leq S+1$ be least such that

$$\mathrm{Th}_{k+1}^{\mathcal{M}}(\alpha_s \cup \{\alpha_0, \cdots, \alpha_{s-1}, u\}) \notin |\mathcal{M}| .$$

Such an s certainly exists, since $S+1$ will do. Let

$$\mathcal{H} = \mathcal{H}_{k+1}^{\mathcal{M}}(\alpha_s \cup \{\alpha_0, \cdots, \alpha_{s-1}, u\}) ,$$

let $\pi : \mathcal{H} \to \mathcal{M}$ be the inverse of the collapse (so that π is a k-embedding), and let $\bar{u} = \pi^{-1}(u)$ and $\bar{\alpha}_j = \pi^{-1}(\alpha_j)$ for $j < s$.

Our strategy is to compare $\mathcal{H}$ with $\mathcal{M}$, using k-maximal trees. Suppose that $\mathcal{P}$ is the model produced at the end on the $\mathcal{H}$ side, and $\mathcal{Q}$ the model produced on the $\mathcal{M}$ side. Suppose the branches $\mathcal{H}$ to $\mathcal{P}$ and $\mathcal{M}$ to $\mathcal{Q}$ involve no dropping of any kind, so that we have generalized $r\Sigma_{k+1}$ maps $i \colon \mathcal{H} \to \mathcal{P}$ and $j \colon \mathcal{M} \to \mathcal{Q}$. Suppose $\operatorname{crit} i \geq \alpha_s$ and $\operatorname{crit} j \geq \rho_{k+1}^{\mathcal{M}}$. Then

$$\mathrm{Th}_{k+1}^{\mathcal{H}}(\alpha_s \cup \{\bar{\alpha}_0, \cdots, \bar{\alpha}_{s-1}, \bar{u}\}) = \mathrm{Th}_{k+1}^{\mathcal{P}}(\alpha_s \cup \{i(\bar{\alpha}_0), \cdots, i(\bar{\alpha}_{s-1}), i(\bar{u})\}) \notin \mathcal{Q}$$