33 Louveau's Theorem

Let us define codes for Borel sets in our usual way of thinking of them as trees with basic clopen sets attached to the terminal nodes.

Definitions

1. Define (T, q) is an α-code iff $T \subseteq \omega^{<\omega}$ is a tree of rank $\leq \alpha$ and $q : T^0 \rightarrow \mathcal{B}$ is a map from the terminal nodes, T^0, of T (i.e. rank zero nodes) to a nice base, $\mathcal{B}$, for the clopen sets of ω^ω, say all sets of the form $[s]$ for $s \in \omega^{<\omega}$ plus the empty set.

2. Define $S^s(T, q)$ and $P^s(T, q)$ for $s \in T$ by induction on the rank of s as follows. For $s \in T^0$ define

$$P^s(T, q) = q(s) \text{ and } S^s(T, q) = \sim q(s).$$

For $s \in T^{>0}$ define

$$P^s(T, q) = \bigcup \{S(T, q)^{s\hat{\ }m} : s\hat{\ }m \in T\} \text{ and } S^s(T, q) = \sim P^s(T, q).$$

3. Define

$$P(T, q) = P^{\langle\rangle}(T, q) \text{ and } S(T, q) = S^{\langle\rangle}(T, q)$$

the Π^0_α set and the Σ^0_α set coded by (T, q), respectively. (S is short for Sigma and P is short for Pi.)

4. Define $C \subseteq \omega^\omega$ is $\Pi^0_\alpha(\text{hyp})$ iff it has an α-code which is hyperarithmetic.

5. ω_1^{CK} is the first nonrecursive ordinal.

Theorem 33.1 *(Louveau [63]) If $A, B \subseteq \omega^\omega$ are Σ^1_1 sets, $\alpha < \omega_1^{CK}$, and A and B can be separated by $\boldsymbol{\Pi}^0_\alpha$ set, then A and B can be separated by a $\Pi^0_\alpha(\text{hyp})$-set.*

Corollary 33.2 $\Delta^1_1 \cap \boldsymbol{\Pi}^0_\alpha = \Pi^0_\alpha(\text{hyp})$

Corollary 33.3 *(Section Problem) If $B \subseteq \omega^\omega \times \omega^\omega$ is Borel and $\alpha < \omega_1$ is such that $B_x \in \boldsymbol{\Sigma}^0_\alpha$ for every $x \in \omega^\omega$, then*

$$B \in \boldsymbol{\Sigma}^0_\alpha(\{D \times C : D \in \text{Borel}(\omega^\omega) \textit{ and } C \textit{is clopen}\}).$$

Note that the converse is trivial.

This result was proved by Dellecherie for $\alpha = 1$ who conjectured it in general. Saint-Raymond proved it for $\alpha = 2$ and Louveau and Saint-Raymond independently proved it for $\alpha = 3$ and then Louveau proved it in general. In their paper [64] Louveau and Saint-Raymond give a different proof of it. We will need the following lemma.

Lemma 33.4 *For $\alpha < \omega_1^{CK}$ the following sets are Δ^1_1:*
$\{y : y$ is a β-code for some $\beta < \alpha\}$,
$\{(x, y) : y$ is a β-code for some $\beta < \alpha$ and $x \in P(T, q)\}$, and
$\{(x, y) : y$ is a β-code for some $\beta < \alpha$ and $x \in S(T, q)\}$.