# 32 $\Sigma^1_1$ equivalence relations

**Theorem 32.1** *(Burgess [14]) Suppose $E$ is a $\mathbf{\Sigma}^1_1$ equivalence relation. Then either $E$ has $\leq \omega_1$ equivalence classes or there exists a perfect set of pairwise $E$-inequivalent reals.*

proof:

We will need to prove the boundedness theorem for this result. Define

$$WF = \{T \subseteq \omega^{<\omega} : T \text{is a well-founded tree}\}.$$

For $\alpha < \omega_1$ define $WF_{<\alpha}$ to the subset of $WF$ of all well-founded trees of rank $< \alpha$. $WF$ is a complete $\Pi^1_1$ set, i.e., for every $B \subseteq \omega^\omega$ which is $\mathbf{\Pi}^1_1$ there exists a continuous map $f$ such that $f^{-1}(WF) = B$ (see Theorem 17.4). Consequently, $WF$ is not Borel. On the other hand each of the $WF_{<\alpha}$ are Borel.

**Lemma 32.2** *For each $\alpha < \omega_1$ the set $WF_{<\alpha}$ is Borel.*

proof:

Define for $s \in \omega^{<\omega}$ and $\alpha < \omega_1$

$$WF^s_{<\alpha} = \{T \subseteq \omega^{<\omega} : T \text{ is a tree, } s \in T,\ \ r_T(s) < \alpha\}.$$

The fact that $WF^s_{<\alpha}$ is Borel is proved by induction on $\alpha$. The set of trees is $\Pi^0_1$. For $\lambda$ a limit

$$WF^s_{<\lambda} = \bigcup_{\alpha<\lambda} WF^s_{<\alpha}.$$

For a successor $\alpha + 1$

$$T \in WF^s_{<\alpha+1} \text{ iff } s \in T \text{ and } \forall n\ \ (s\hat{\ }n \in T \rightarrow T \in WF^{s\hat{\ }n}_{<\alpha}).$$

■

Another way to prove this is take a tree $T$ of rank $\alpha$ and note that $WF_{<\alpha} = \{\hat{T} : \hat{T} \prec T\}$ and this set is $\mathbf{\Delta}^1_1$ and hence Borel by Theorem 26.1.

**Lemma 32.3** *(Boundedness) If $A \subseteq WF$ is $\mathbf{\Sigma}^1_1$, then there exists $\alpha < \omega_1$ such that $A \subseteq WF_\alpha$.*

proof:

Suppose no such $\alpha$ exists. Then

$$T \in WF \text{ iff there exists } \hat{T} \in A \text{ such that } T \preceq \hat{T}.$$

But this would give a $\mathbf{\Sigma}^1_1$ definition of $WF$, contradiction.

■

There is also a lightface version of the boundedness theorem, i.e., if $A$ is a $\Sigma^1_1$ subset of $WF$, then there exists a recursive ordinal $\alpha < \omega_1^{CK}$ such that $A \subseteq WF_{<\alpha}$. Otherwise,