Differential algebraic groups and the number of countable differentially closed fields

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Introduction.

We give an exposition of several results on definable groups in differentially closed fields, and applications thereof. Among other things we give a proof of the result [HS] that there are continuum many countable differentially closed fields of characteristic 0. The theory DCF_0 (of differentially closed fields of characteristic 0) is complete and ω-stable and thus by [SHM] has either $\leq \aleph_0$ or $2^{\aleph_0}$ countable models. But until recently it was not known which. Rather surprisingly it turns out that classical mathematical objects, specifically elliptic curves, lie behind the existence of continuum many countable models (or at least behind the present proof). One of the essential points is to find some strongly regular nonisolated type which is orthogonal to the empty set. The required type p is found inside a suitable definable (in DCF_0) subgroup G (of finite Morley rank) of an elliptic curve $E(a)$ with differentially transcendental j-invariant a. So it turns out that there are "exotic" groups of finite Morley rank definable in differentially closed fields. In any case in section 2 of this paper we prove the existence of $2^{\aleph_0}$ countable differentially closed fields. The argument we present was sketched for us by E. Hrushovski, although we have a few additional simplifications. In fact, given an example due to Manin [M], showing that for any elliptic curve E there is differential rational homomorphism from E onto G_a (the additive group), the existence of the required type p turns out to rather a direct matter, requiring neither the deep Zariski-geometry interpretation, nor the properties of "jet groups" of algebraic groups.

On the other hand, in so far as simple (noncommutative) groups of finite Morley rank are concerned, no exotic structures are to be found in differentially closed fields. Any such group G will be definably isomorphic to an algebraic group living in the constants. This is exactly the finite Morley rank case of Cassidy's Theorem [C2], of which I will give an easy proof in section 1. This implies that any infinite field F of finite Morley rank definable in a differentially closed field K is definably isomorphic to the field of constants of K. The remaining part (namely the infinite Morley rank case) of Cassidy's Theorem, states that a simple group of infinite Morley rank definable in a differentially closed field K is definably isomorphic to an algebraic group over K. We were unable to find