Model Theory of Differential Fields

David Marker
University of Illinois at Chicago

§1 Differential Algebra.

Throughout these notes *ring* will mean commutative ring with identity.

A *derivation* on a ring R is an additive homomorphism $D : R \to R$ such that $D(xy) = xD(y) + yD(x)$. A *differential ring* is a ring equipped with a derivation.

Derivations satisfy all of the usual rules for derivatives. Let D be a derivation on R.

Lemma 1.1. For all $x \in R$, $D(x^n) = nx^{n-1}D(x)$.

Proof.
By induction on n. $D(x^1) = D(x)$.

$$\begin{aligned} D(x^{n+1}) &= D(xx^n) = xD(x^n) + x^n D(x) \\ &= nx^n D(x) + x^n D(x) \\ &= (n+1)x^n D(x). \end{aligned}$$

Lemma 1.2. If b is a unit of R, $D(\frac{a}{b}) = \frac{bD(a)-aD(b)}{b^2}$.

Proof.
$D(a) = D(b \cdot \frac{a}{b}) = bD(\frac{a}{b}) + \frac{a}{b}D(b)$.
Thus $D(\frac{a}{b}) = \frac{1}{b}D(a) - \frac{a}{b^2}D(b) = \frac{bD(a)-aD(b)}{b^2}$.

examples.
1) (trivial derivation) $D : R \to \{0\}$.
2) Let C^∞ be the ring of infinitely differentiable real functions on $(0,1)$ and let D be the usual derivative.
3) Let U be a nonempty connected open subset of $\mathbf{C}$. Let O_U be the ring of analytic functions $f : U \to \mathbf{C}$ and let $D : O_U \to O_U$ be the usual derivative. [Note: O_U is an integral domain, while the ring of C^∞ functions is not.] Similarly the field of meromorphic functions on U is a differential field. In appendix A, we show that every countable differential field can be embedded into a field of germs of meromorphic functions.
4) Let $a \in R$. Let $D : R[X] \to R[X]$ by $D(\sum a_i X^i) = a(\sum i a_i X^{i-1})$. [Note: If $a = 1$, then D is $\frac{d}{dX}$.]