§1 Differential Algebra.

Throughout these notes ring will mean commutative ring with identity.

A derivation on a ring \( R \) is an additive homomorphism \( D : R \rightarrow R \) such that \( D(xy) =xD(y)+yD(x) \). A differential ring is a ring equipped with a derivation.

Derivations satisfy all of the usual rules for derivatives. Let \( D \) be a derivation on \( R \).

**Lemma 1.1.** For all \( x \in R \), \( D(x^n) = nx^{n-1}D(x) \).

**Proof.**

By induction on \( n \). \( \text{D}(x^1) = D(x) \).

\[
\text{D}(x^{n+1}) = \text{D}(xx^n) = x\text{D}(x^n) + x^n\text{D}(x) = nx^n\text{D}(x) + x^n\text{D}(x) = (n+1)x^n\text{D}(x).
\]

**Lemma 1.2.** If \( b \) is a unit of \( R \), \( D\left(\frac{a}{b}\right) = \frac{bD(a) - aD(b)}{b^2} \).

**Proof.**

\[
D(a) = D(b \cdot \frac{a}{b}) = bD\left(\frac{a}{b}\right) + \frac{a}{b}D(b).
\]

Thus \( D\left(\frac{a}{b}\right) = \frac{1}{b}D(a) - \frac{a}{b^2}D(b) = \frac{bD(a) - aD(b)}{b^2} \).

**Examples.**

1) (trivial derivation) \( D : R \rightarrow \{0\} \).

2) Let \( C^{\infty} \) be the ring of infinitely differentiable real functions on \((0,1)\) and let \( D \) be the usual derivative.

3) Let \( U \) be a nonempty connected open subset of \( \mathbb{C} \). Let \( O_U \) be the ring of analytic functions \( f : U \rightarrow \mathbb{C} \) and let \( D : O_U \rightarrow O_U \) be the usual derivative. [Note: \( O_U \) is an integral domain, while the ring of \( C^{\infty} \) functions is not.] Similarly the field of meromorphic functions on \( U \) is a differential field. In appendix A, we show that every countable differential field can be embedded into a field of germs of meromorphic functions.

4) Let \( a \in R \). Let \( D : R[X] \rightarrow R[X] \) by \( D(\sum a_iX^i) = a(\sum ia_iX^{i-1}) \). [Note: If \( a = 1 \), then \( D \) is \( \frac{d}{dX} \).]