Introduction to the Model Theory of Fields

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My goal in these lectures is to survey some classical and recent results in model theoretic algebra. We will concentrate on the fields of real and complex numbers and discuss connections to pure model theory and algebraic geometry.

Our basic language will be the language of rings $\mathcal{L}_{\mathrm{r}} = \{+,-,\cdot,0,1\}$. The field axioms, T_{fields}, consists of the universal axioms for integral domains and the axiom
$\forall x \exists y\ (x = 0 \vee xy = 1)$. Since every integral domain can be extended to its fraction field, integral domains are exactly the $\mathcal{L}_{\mathrm{r}}$-substructures of fields. For a fixed field F we will study the subsets of F^n which are defined in the language $\mathcal{L}_{\mathrm{r}}$.

§1 Algebraically closed fields

Let ACF be T_{fields} together with the axiom

$$\forall a_0 \ldots \forall a_{n-1} \exists x\ x^n + \sum_{i=0}^{n-1} a_i x^i = 0$$

for each n. Clearly ACF is not a complete theory since it does not decide the characteristic of the field. For each n let ϕ_n be the formula

$$\forall x\ \underbrace{x + \ldots + x}_{n \text{ times}} = 0.$$

For p prime, let ACF_p be theory $ACF + \phi_p$, and let $ACF_0 = ACF \cup \{\neg\phi_n : n = 1,2,\ldots\}$.

For our purposes the key algebraic fact about algebraically closed fields is that they are described up to isomorphism by the characteristic and the transcendence degree. This has important model theoretic consequences. Recall that for a cardinal κ a theory is *κ-categorical* if there is, up to isomorphism, a unique model of cardinality κ.

Proposition 1.1. Let p be prime or zero and let κ be an uncountable cardinal. The theory ACF_p is κ-categorical, complete, and decidable.