A Bounded Arithmetic Theory for Constant Depth Threshold Circuits*

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Summary. We define an extension $\bar{R}_2^0$ of the bounded arithmetic theory R_2^0 and show that the class of functions Σ_1^b-definable in $\bar{R}_2^0$ coincides with the computational complexity class TC^0 of functions computable by polynomial size, constant depth threshold circuits.

1. Introduction

The theories S_2^i, for $i \in \mathrm{N}$, of Bounded Arithmetic were introduced by Buss [3]. The language of these theories is the language of Peano Arithmetic extended by symbols for the functions $\lfloor \frac{1}{2}x \rfloor$, $|x| := \lceil \log_2(x+1) \rceil$ and $x \# y := 2^{|x| \cdot |y|}$. A quantifier of the form $\forall x \leq t$, $\exists x \leq t$ with x not occurring in t is called a *bounded quantifier*. Furthermore, a quantifier of the form $\forall x \leq |t|$, $\exists x \leq |t|$ is called *sharply bounded.* A formula is called (sharply) bounded if all quantifiers in it are (sharply) bounded.

The class of bounded formulae is divided into an hierarchy analogous to the arithmetical hierarchy: The class of sharply bounded formulae is denoted Σ_0^b or Π_0^b. For $i \in \mathrm{N}$, Σ_{i+1}^b (resp. Π_{i+1}^b) is the least class containing Π_i^b (resp. Σ_i^b) and closed under conjunction, disjunction, sharply bounded quantification and bounded existential (resp. universal) quantification.

Now the theory S_2^i is defined by a finite set $BASIC$ of quantifier-free axioms plus the scheme of *polynomial induction*

$$A(0) \wedge \forall x \,(A(\lfloor \frac{1}{2} x \rfloor) \rightarrow A(x)) \;\rightarrow\; \forall x A(x)$$

for every Σ_i^b-formula $A(x)$ (Σ_i^b-$PIND$).

For a class of formulae Γ, a number-theoretic function f is said to be Γ-definable in a theory T if there is a formula $A(\bar{x}, y) \in \Gamma$, describing the graph of f in the standard model, and a term $t(\bar{x})$, such that T proves

$$\forall \bar{x} \; \exists y \leq t(\bar{x}) \; A(\bar{x}, y)$$

$$\forall \bar{x}, y_1, y_2 \; A(\bar{x}, y_1) \wedge A(\bar{x}, y_2) \rightarrow y_1 = y_2$$

The main result of [3] relates the theories S_2^i to the Polynomial Time Hierarchy PH of Computational Complexity Theory (cf. [9]):

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