Infinite-valued Gödel Logics with 0-1-Projections and Relativizations *

Matthias Baaz **

Institut für Algebra und Diskrete Mathematik E118.2,
Technische Universität Wien, A–1040 Vienna, Austria
Email address: baaz@logic.tuwien.ac.at

Summary. Infinite-valued Gödel logic, i.e., Dummett's **LC**, is extended by projection modalities and relativizations to truth value sets. An axiomatization for the corresponding propositional logic (sound and complete relative to any infinite set of truth values) is given. It is shown that certain simple infinite sets of truth values correspond to first-order Gödel logics which are not recursively axiomatizable.

1. Introduction

One of Gödel's main contributions to the study of nonstandard, in particular, many-valued and intuitionistic logics was his [4]. In that paper, he introduced a sequence of finite-valued propositional logics $\mathbf{G}_n$ intermediate in strength between classical and intuitionistic propositional logic. The definition of $\mathbf{G}_n$ is uniform, i.e., makes no explicit reference to the number of truth values. The only restrictions on the set of truth values V are that V is a (linearly ordered) subset of $[0, 1]$ and that $0, 1 \in V$. Dummett [3] subsequently showed that the infinite-valued Gödel logics are axiomatized by intuitionistic propositional calculus plus the axiom schema $(A \supset B) \vee (B \supset A)$. We extend this result to infinite-valued Gödel logics with the projection modalities on 0 and 1:

$$\nabla(A) = \begin{cases} 1 & \text{if } A \neq 0 \\ 0 & \text{if } A = 0 \end{cases} \qquad \triangle(A) = \begin{cases} 1 & \text{if } A = 1 \\ 0 & \text{if } A \neq 1 \end{cases}$$

Only the addition of $\triangle$ is of interest, as ∇ may be defined by $\nabla(A) \equiv \neg\neg A$.

The main result of the first part of this paper is the completeness theorem for all infinite sets of truth values V for the axiomatization consisting of the axiom schemas of intuitionistic propositional logic and of modal logic **S4** for $\triangle$ (including the necessitation rule $A/\triangle A$), plus the following schemas:

$$(A \supset B) \vee (B \supset A)$$
$$\triangle A \vee \neg\triangle A$$
$$\triangle(A \vee B) \supset \triangle A \vee \triangle B$$

* This paper is in its final form and no similar paper has been or is being submitted elsewhere.

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