CHAPTER 3

APPLICATIONS TO PROTOALGEBRAIC AND ALGEBRAIZABLE LOGICS

One of the most important classes of sentential logics from the point of view of their algebraization is the class of the protoalgebraic logics. As defined in Blok and Pigozzi [1986], a sentential logic is ***protoalgebraic*** when for any $\mathit{\Gamma} \in \mathcal{T}h\mathcal{S}$, any two formulas equivalent modulo $\boldsymbol{\Omega}_{\boldsymbol{Fm}}(\mathit{\Gamma})$ are also $\mathcal{S}$-interderivable modulo $\mathit{\Gamma}$; that is, when for any $\mathit{\Gamma} \in \mathcal{T}h\mathcal{S}$ and any $\varphi, \psi \in Fm$,

$$\text{if } \langle \varphi, \psi \rangle \in \boldsymbol{\Omega}_{\boldsymbol{Fm}}(\mathit{\Gamma}) \quad \text{then} \quad \mathit{\Gamma}, \varphi \vdash_{\mathcal{S}} \psi \text{ and } \mathit{\Gamma}, \psi \vdash_{\mathcal{S}} \varphi,$$

or, in our notation, when for any $\mathit{\Gamma} \in \mathcal{T}h\mathcal{S}$, $\boldsymbol{\Omega}_{\boldsymbol{Fm}}(\mathit{\Gamma}) \subseteq \boldsymbol{\Lambda}_{\mathcal{S}}(\mathit{\Gamma})$.

This class of logics was defined and thoroughly studied in Blok and Pigozzi [1986]. Independently, it was considered in Czelakowski [1985], with a different definition and under the name of *non-pathological logics*; the equivalence of the two definitions was proved in Blok and Pigozzi [1992]. From the results in these and subsequent works (such as Blok and Pigozzi [1991], Czelakowski [2001a] and Czelakowski and Dziobiak [1991]) one can reach the conclusion that these logics are precisely the ones whose matrix semantics is particularly well-behaved from the point of view of universal algebra. Among several interesting characterizations of this notion, let us mention that a logic $\mathcal{S}$ is protoalgebraic iff the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$ on $\mathcal{T}h\mathcal{S}$ is monotone with respect to $\subseteq$. This is also equivalent to saying that for any algebra $\boldsymbol{A}$, the operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is monotone on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ (see Blok and Pigozzi [1986] Theorem 2.4); this property is called the *Compatibility Property*. Let us look more closely into what this property says: Being monotone means that for any $\boldsymbol{A}$ and any $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $F \subseteq G$ then $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(G)$. Observe that $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(G)$ is equivalent to saying that $\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ is compatible with G, that is, that G is a union of equivalence classes modulo $\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$; if we consider the canonical projection $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$, another way of expressing the compatibility property is to say that $G = \pi^{-1}\big[\pi[G]\big]$ for all $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$. Taking Proposition 1.19 into account,