## §6. An inductive definition of $K$

The definition of $K$ given in 5.17 is $\Sigma_\omega(V_{\Omega+1})$, and therefore much too complicated for some purposes. In this section we shall give an inductive definition of $K$ whose logical form is as simple as possible. Assuming that $K^c$ has no Woodin cardinals, we shall show that $K \cap HC$ is $\Sigma_1(L_{\omega_1}(\mathbb{R}))$ in the codes; Woodin has shown that in general no simpler definition is possible.

The following notion is central to our inductive definition of $K$.

**Definition 6.1.** *Let $\mathcal{M}$ be a proper premouse such that $\mathcal{M} \models ZF - \{Powerset\}$ and $\mathcal{J}_\alpha^{\mathcal{M}}$ is $S$-sound. We say $\mathcal{M}$ is $(\alpha, S)$-strong iff there is an $(\omega, \Omega+1)$ iterable weasel which witnesses that $\mathcal{J}_\alpha^{\mathcal{M}}$ is $S$-sound, and whenever $W$ is a weasel which witnesses that $\mathcal{J}_\alpha^{\mathcal{M}}$ is $S$-sound, and $\Sigma$ is an $(\omega, \Omega+1)$ iteration strategy for $W$, then there is a length $\theta+1$ iteration tree $\mathcal{T}$ on $W$ which is a play by $\Sigma$ and such that $\forall\gamma < \theta(\nu(E_\gamma^{\mathcal{T}}) \geq \alpha)$, and a $Q \trianglelefteq W_\theta^{\mathcal{T}}$, and a fully elementary $\pi : \mathcal{M} \to Q$ such that $\pi \restriction \alpha =$ identity.*

We shall see that it is possible to define "$(\alpha, S)$-strong" by induction on $\alpha$. First, let us notice:

**Lemma 6.2.** *Let $W$ be an $(\omega, \Omega+1)$ iterable weasel which witnesses that $\mathcal{J}_\alpha^W$ is $S$-sound; then $W$ is $(\alpha, S)$ strong.*

*Proof.* Let $R$ be a weasel which witnesses $\mathcal{J}_\alpha^W$ is $S$-sound, and let $\Sigma$ be an $\Omega+1$ iteration strategy for $R$. Let $\Gamma$ be an $\Omega+1$ iteration strategy for $W$, and let $(\mathcal{T}, \mathcal{U})$ be the successful coiteration of $R$ with $W$ determined by $(\Sigma, \Gamma)$. Let $Q$ be the common last model of $\mathcal{T}$ and $\mathcal{U}$, and let $\pi : W \to Q$ be the iteration map given by $\mathcal{U}$. By Lemma 5.1, $\pi \restriction \alpha =$ identity. □

Lemma 6.2 admits the following slight improvement. Let $W$ witness that $\mathcal{J}_\alpha^W$ is $S$-sound, and let $\Sigma$ be an $(\omega, \Omega+1)$ iteration strategy for $W$. Let $\mathcal{T}$ be an iteration tree played by $\Sigma$ such that $\forall\gamma < \theta(\nu(E_\gamma^{\mathcal{T}}) \geq \alpha)$, where $\theta+1 = lh\ \mathcal{T}$; then $W_\theta^{\mathcal{T}}$ is $(\alpha, S)$ strong. [Proof: Let $R$ be any weasel witnessing $\mathcal{J}_\alpha^W$ is $S$-sound. Comparing $R$ with $W$, we get an iteration tree $\mathcal{U}$ on $R$ and a map $\pi : W \to R_\eta^{\mathcal{U}}$, where $\eta = lh\,\mathcal{U} - 1$. By 5.1, $\text{crit}(\pi) \geq \alpha$. Let $\sigma : W_\theta^{\mathcal{T}} \to (R_\eta^{\mathcal{U}})_\theta^{\pi\mathcal{T}}$ be the copy map. Then $\sigma$ and $\mathcal{U}^\frown \pi\mathcal{T}$ are as required in 6.1 for $R$.] This shows that we obtain a definition of $(\alpha, S)$ strength equivalent to 6.1 if we replace "whenever $W$ is a weasel" by "there is a weasel $W$" in 6.1. It also shows that there are $(\alpha, S)$ strong weasels other than those described in 6.2. For example, suppose $W$ witnesses that $\mathcal{J}_\alpha^W$ is $S$-sound, and $E$ is an extender on the $W$ sequence which is total on $W$ and such that $\text{crit}(E) < \alpha \leq \nu(E)$. Setting $R = \text{Ult}(W, E)$, we have that $R$ is $(\alpha, S)$ strong, but $R$ does not witness that $\mathcal{J}_\alpha^R$ is $S$-sound.

In view of the fact that $K(S)$ is independent of $S$, one might expect the same to be true of $(\alpha, S)$-strength. This is indeed the case.