7. Canonization for Two Variables

In this chapter we prove that both $L^2_{\infty\omega}$ and $C^2_{\infty\omega}$ admit PTIME canonization. We do so by exhibiting PTIME inverses for I_{L^2} and I_{C^2}. The inversion for I_{L^2} is even PTIME in terms of the size of the I_{L^2}, a phenomenon that we know to be peculiar to the two variable case. These are the main theorems:

Theorem 7.1. *I_{L^2} admits* PTIME *inversion in the strong sense that for each finite relational τ there is a* PTIME *functor $F\colon \{I_{L^2}(\mathfrak{A}) \mid \mathfrak{A} \in \mathrm{fin}[\tau]\} \to \mathrm{stan}[\tau]$, which is an inverse for I_{L^2}:*

$$\forall \mathfrak{A} \quad F\big(I_{L^2}\mathfrak{A}\big) \equiv^{L^2} \mathfrak{A}.$$

It follows that

(i) the range of I_{L^2} can be recognized in PTIME.
(ii) $L^2_{\infty\omega}$ admits PTIME *canonization.*
(iii) PTIME $\cap\, L^2_{\infty\omega}$ *is recursively enumerable (has a recursive presentation).*
(iv) PTIME $\cap\, L^2_{\infty\omega} \equiv \mathrm{FP}\big(I_{L^2}\big) \equiv \mathrm{PTIME}\big(I_{L^2}\big)$.

Compare the general Theorems 6.11 and 6.14 for (ii) and (iii). (i) is obvious: for $\mathfrak{I}$ of the format of an L^2-invariant, $\mathfrak{I} \in \{I_{L^2}(\mathfrak{A}) \mid \mathfrak{A} \in \mathrm{fin}[\tau]\}$ if and only if $F(\mathfrak{I}) \in \mathrm{fin}[\tau]$ and $I_{L^2}\big(F(\mathfrak{I})\big) = \mathfrak{I}$. (i) and the strong form of (iv) (if compared to the statement of Theorem 6.14) are consequences of polynomiality of F in the usual sense.

Theorem 7.2. *I_{C^2} admits* PTIME *inversion. For each finite relational τ there is a* PTIME *functor $F\colon \{I_{C^2}(\mathfrak{A}) \mid \mathfrak{A} \in \mathrm{fin}[\tau]\} \to \mathrm{stan}[\tau]$, which is an inverse for I_{C^2}:*

$$\forall \mathfrak{A} \quad F\big(I_{C^2}\mathfrak{A}\big) \equiv^{C^2} \mathfrak{A}.$$

It follows that

(i) the range of I_{C^2} can be recognized in PTIME.
(ii) $C^2_{\infty\omega}$ admits PTIME *canonization.*
(iii) PTIME $\cap\, C^2_{\infty\omega}$ *is recursively enumerable (has a recursive presentation).*
(iv) PTIME $\cap\, C^2_{\infty\omega} \equiv \mathrm{FP}\big(I_{C^2}\big) \equiv \mathrm{PTIME}\big(I_{C^2}\big)$.