4. Fixed-Point Logic with Counting

This chapter is devoted to the introduction and analysis of the natural extensions of the fixed-point logics FP and PFP that have expressive means for cardinality properties.

• The actual formalization of fixed-point logics with counting, FP+C and PFP+C, in a two-sorted framework is given in Section 4.1.

• In Section 4.2 the relation of FP+C and PFP+C with the $C^k_{\infty\omega}$ and with the C^k-invariants is investigated. In particular we obtain the analogue of the first theorem of Abiteboul and Vianu (Theorem 3.22 above) in the presence of counting. In contrast with the second theorem of Abiteboul and Vianu (Theorem 3.24) we here find that FP+C *is* the polynomial restriction of PFP+C.

• Section 4.3 deals with the separation result FP+C $\subsetneq$ PTIME, which is due to Cai, Fürer and Immerman, in a framework that lends itself to relativization. In restriction to classes with certain closure properties FP+C can only capture PTIME if some I_{C^k} provides a complete invariant *up to isomorphism* (equivalently, if some $C^k_{\infty\omega}$ coincides with $L_{\infty\omega}$) over this class.

• Section 4.4 summarizes some results on equivalent characterizations of the expressive levels of FP+C and PFP+C.

As pointed out in the introduction, first-order logic at first sight suffers from *two* independent shortcomings over finite structures: it completely lacks mechanisms to model recursion — the fixed-point operations provided in FP and PFP answer this requirement; and it also lacks expressive means to assess cardinalities of definable sets. The latter defect is obviously overcome automatically together with the former over ordered structures. By the theorems of Immerman, Vardi and Abiteboul, Vardi, Vianu, FP and PFP capture PTIME and PSPACE over ordered structures. In particular all PTIME, respectively PSPACE, properties of cardinalities are expressible in FP, respectively PFP, over ordered structures. Not so in the case of not necessarily ordered structures: in fact the most obvious examples that FP and PFP do not correspond to standard complexity classes in the general case all involve counting. Over pure sets for instance FP, PFP and even $L^\omega_{\infty\omega}$ collapse to first-order