2. The Games and Their Analysis

This chapter serves to review the Ehrenfeucht-Fraïssé style analysis of the logics $L^k_{\infty\omega}$ and $C^k_{\infty\omega}$ by means of the corresponding *pebble games*. Emphasis is on the games and their algebraic analysis rather than on the more syntactic descriptions in terms of Hintikka formulae and Scott sentences. The main result of this algebraic analysis is a definable *ordering with respect to types.* We obtain ordered representations of the quotients $\mathrm{Tp}^{\mathcal{L}}(\mathfrak{A}; k) = A^k / \equiv^{\mathcal{L}}$ for $\mathcal{L} = L^k_{\infty\omega}$ or $C^k_{\infty\omega}$ on finite relational structures $\mathfrak{A}$.

• Section 2.1 contains the definition of the games and the statement and proofs of the corresponding Ehrenfeucht-Fraïssé theorems which here are due to Barwise [Bar77], Immerman [Imm82], and Immerman and Lander [IL90], respectively. We present some typical examples that apply the game characterizations to derive non-expressibility results. Most notably a construction due to Cai, Fürer and Immerman proves that the logics $C^k_{\infty\omega}$ form a strict hierarchy with respect to k.

A refined analysis of the games shows that $\equiv^{C^k_{\infty\omega}}$ and $\equiv^{C^k_{\omega\omega}}$, and similarly $\equiv^{L^k_{\infty\omega}}$ and $\equiv^{L^k_{\omega\omega}}$, coincide in restriction to finite structures.

• In Section 2.2 we review the colour refinement technique for graphs and discuss some variants and their definability properties.

• Ideas related to the colour refinement are employed in Section 2.3 to introduce the ordered quotients with respect to $C^k_{\infty\omega}$- or $L^k_{\infty\omega}$-types through a fixed-point process for the classification of game positions.

2.1 The Pebble Games for $L^k_{\infty\omega}$ and $C^k_{\infty\omega}$

The setting for the games is the usual one for comparison games. There are two players denoted **I** and **II** for *first* and *second player.* The game is played on a pair of finite structures $\mathfrak{A}$ and $\mathfrak{A}'$ of the same finite relational vocabulary τ. In the *k-pebble game* there are k marked pebbles for each of the two structures. Let both sets of pebbles be numbered $1, \ldots, k$. A *stage* of the game, or an instantaneous description of a game situation, is determined by a placement of the pebbles on elements of the corresponding structures.