8. GENERALIZATIONS

So far our results have been explicitly stated (and proved) only for theories of first order arithmetic. But, as mentioned in the introduction, they hold, after suitable reformulation, in a much more general setting. Needless to say, we are not going to show this in every detail. In fact, we shall skip Chapters 3, 5, 7 altogether and concentrate on some of the main results of Chapters 2, 4, and 6. These examples should enable the reader to generalize (most of) the results of the preceding chapters.

In this chapter he theories S, T, etc. are no longer arithmetical theories, but they are still consistent and primitive recursive and we assume that the languages of these theories are always finite. L_T is the language of T. T is a *pure* extension of S if $S \dashv T$ and $L_T = L_S$. Lower case Greek letters are now used for formulas of L_T as well as for formulas of L_A.

We assume that the reader can extend the definition of t: $S \leq T$ to the present more general setting. Let $t^{-1}(T) = \{\varphi\colon T \vdash t(\varphi)\}$. Then $t^{-1}(T) \vdash \psi$ iff $T \vdash t(\psi)$. Since L_S is finite, t is primitive recursive.

The following lemma is immediate.

Lemma 1. (a) t: $S \leq T$ iff $S \dashv t^{-1}(T)$.

(b) t: $t^{-1}(T) \leq T$ and so $t^{-1}(T) \leq T$; in fact, t: $t^{-1}(T) \trianglelefteq T$; it follows that $t^{-1}(T)$ is consistent.

(c) $t^{-1}(T + t(\varphi)) \dashv\vdash t^{-1}(T) + \varphi$.

§1. Incompleteness. Our first result, Gödel's incompleteness theorem, is a straightforward generalization of Theorem 2.1; $\delta_t(x,y)$ is a formula defining t as in Fact 2.

Theorem 1. Suppose t: $Q \leq T$. Let φ be such that

(Gt) $Q \vdash \varphi \leftrightarrow \neg\exists y(\delta_t(\varphi,y) \wedge Pr_T(y))$.

Then φ is a true Π_1 sentence such that $T \nvdash t(\varphi)$. Hence if $t^{-1}(T)$ is Σ_1–sound, then also $T \nvdash \neg t(\varphi)$.

By Theorem 1, for each t: $Q \leq T$, there is a true Π_1 sentence φ_t such that $T \nvdash t(\varphi_t)$. By a similar generalization of Rosser's theorem, we obtain a Π_1 sentence θ_t such that $T \nvdash t(\theta_t)$ and $T \nvdash \neg t(\theta_t)$. This result can be improved by showing that there is a single Π_1 sentence ψ such that $T \nvdash t(\psi)$ and $T \nvdash \neg t(\psi)$ for every t: $Q \leq T$:

Theorem 2. There is a (true) Π_1 sentence ψ, such that $Q + \psi \not\leq T$ and $Q + \neg\psi \not\leq T$.