7. DEGREES OF INTERPRETABILITY

Suppose PA$\dashv$ T. We shall use A, B, etc. for extensions of T. (Thus, T, A, B, etc. are essentially reflexive.) The relation $\leq$ of interpretability is reflexive and transitive. Thus, the relation $\equiv$ of mutual interpretability (restricted to extensions of T) is an equivalence relation; its equivalence classes will be called *degrees* (*of interpretability*) and will be written a, b, c, etc. D_T is the set of degrees of extensions of T. A is of degree a if A$\in$a and d(A) is the degree of A. The relation $\leq$ among degrees is the relation induced by the relation $\leq$ among theories: d(A) $\leq$ d(B) iff A $\leq$ B. $\mathbf{D}_T = (D_T, \leq)$, the partially ordered set of degrees defined in this way, will be studied in some detail in this chapter.

§1. Algebraic properties. In this § we restrict ourselves to purely algebraic properties of $\mathbf{D}_T$. First we define the theory A^T and the operations $\downarrow$ and $\uparrow$ on theories as follows.

$A^T = T + \{Con_{A\restriction k}\colon k\in N\}$,
$A{\downarrow}B = T + \{Con_{A\restriction k} \vee Con_{B\restriction k}\colon k\in N\}$,
$A{\uparrow}B = T + \{Con_{A\restriction k} \wedge Con_{B\restriction k}\colon k\in N\}$.

From Lemma 6.2 and Theorem 6.6, we get the following:

Lemma 1. (a) A $\leq$ B iff $A^T \dashv$ B. Thus, $A^T \equiv$ A and A $\leq$ B iff $A^T \dashv B^T$.
(b) A $\leq$ B, C iff A $\leq$ B$\downarrow$C,
(c) A, B $\leq$ C iff A$\uparrow$B $\leq$ C.

The following lemma is little more than a restatement of Lemma 4.4.

Lemma 2. If θ is Π_1 and A$\vdash$ θ, there is a k such that PA$\vdash$ $Con_{A\restriction k} \to \theta$.

Instead of A$\downarrow$B it is sometimes convenient to use the theory A$\vee$B defined by

$A\vee B = \{\varphi \vee \psi\colon \varphi\in A\ \&\ \psi\in B\}$.

Th(A$\vee$B) = Th(A) $\cap$ Th(B). Evidently, A$\downarrow$B $\dashv$ A$\vee$B and, by Lemma 2, A$\vee$B $\dashv_{\Pi_1}$A$\downarrow$B. But then, by Theorem 6.6, that A$\vee$B $\leq$ A$\downarrow$B and so A$\vee$B $\equiv$ A$\downarrow$B. It follows that for every sentence φ, $(A + \varphi){\downarrow}(A + \neg\varphi) \leq A$.

From Lemma 2 and Lemma 6.1 we get:

Lemma 3. For every Π_1 sentence π, T + $\pi \leq$ A$\uparrow$B iff A$\uparrow$B$\vdash$ π iff there are Π_1 sentences φ, ψ such that A$\vdash$ φ, B$\vdash$ ψ, and T + $\varphi \wedge \psi\vdash$ π.

For A$\in$a and B$\in$b, let a $\cap$ b = d(A$\downarrow$B) and a $\cup$ b = d(A$\uparrow$B). By Lemma 1, $\cap$ and $\cup$