4. AXIOMATIZATIONS

S is an *axiomatization* of T if $S \dashv\vdash T$. Suppose $S \dashv T$. $S + X$ is an *axiomatization of* T *over* S if X is r.e. and $T \dashv\vdash S + X$. In this chapter we discuss some important properties of axiomatizations: finiteness, boundedness, and irredundance.

§1. Finite and bounded axiomatizability; reflection principles. We shall say that T is a *finite extension of* S if there is a sentence φ such that $T \dashv\vdash S + \varphi$. T is *essentially infinite over* S if no consistent extension of T is finite over S. T is *essentially infinite* if T is essentially infinite over the empty theory (logic). We already know that PA is essentially infinite (Corollary 2.1).

By the *local reflection principle for* S we understand the set

$$\mathrm{Rfn}_S = \{\mathrm{Pr}_S(\varphi) \to \varphi\colon \varphi \text{ any sentence of } L_A\}.$$

Thus, Rfn_S is a piecemeal (local) way of saying that every sentence provable in S is true. (The latter statement, the full (global) reflection principle for S, cannot be expressed in T, since, by the Gödel–Tarski theorem, truth is not definable.)

Clearly $PA + \mathrm{Rfn}_T \vdash \mathrm{Con}_T$ (let $\varphi := \bot$). Also note that T is essentially reflexive iff $T \vdash \mathrm{Rfn}_{T{\restriction}k}$ for every k (cf. Corollary 1.9 (b)).

We now use the local reflection principle to construct an essentially infinite extension of a given theory S. Note that $\mathrm{Rfn}_S \dashv T$ implies $S \dashv T$.

Theorem 1. If $\mathrm{Rfn}_S \dashv T$, then T is essentially infinite over S.

Proof. Suppose $T \dashv S + \theta$. We are going to show that $S + \theta$ is inconsistent. Let ψ be such that

(1) $\quad Q \vdash \psi \leftrightarrow \neg\mathrm{Pr}_{S+\theta}(\psi)$.

By hypothesis,

$$T \vdash \mathrm{Pr}_S(\theta \to \psi) \to (\theta \to \psi).$$

From this and (1) it follows that $T \vdash \theta \to \psi$. But then

(2) $\quad S + \theta \vdash \psi$.

It follows that $Q \vdash \mathrm{Pr}_{S+\theta}(\psi)$ and so, by (1), $Q \vdash \neg\psi$. But $Q \dashv S + \theta$ and so, by (2), $S + \theta$ is inconsistent. ■

If $PA \dashv T$, the conclusion of Theorem 1 can be strengthened; see Corollary 2, below.

There is a stronger principle, the *uniform reflection principle*, which is a better approximation than Rfn_S of the full reflection principle for S, namely,

$$\mathrm{RFN}_S = \{\forall x(\Gamma(x) \wedge \mathrm{Pr}_S(x) \to \mathrm{Tr}_\Gamma(x))\colon \Gamma \text{ arbitrary}\}.$$

Clearly $T + \mathrm{RFN}_S \vdash \mathrm{Rfn}_S$ provided that $PA \dashv T$. Applying the uniform reflection principle we can derive a stronger conclusion than in Theorem 1.

A set X of sentences is *bounded* if $X \subseteq \Gamma$ for some Γ. Let $\mathrm{Prf}_{S,\Gamma}(x,y) :=$