3. NUMERATIONS OF R.E. SETS

Any set numerated in T is r.e. The question arises if the converse of this is true, in other words, if every r.e. set can be numerated in T. If T is Σ_1–sound, then, of course, the answer is "yes" (Corollary 1.4). If T is not Σ_1–sound, the answer is still "yes" although this is not so obvious. This is the first and most important result of this chapter. We also prove some refinements of this result.

Beginning in this chapter we omit most references to the Lemmas, Facts, and Corollaries of Chapter 1. To avoid too much repetition, proofs are sometimes left to the reader.

§1. Numerations of r.e. sets. Let X be any r.e. set. Our first task is to show that X can be numerated in T even if T is not Σ_1–sound. We have already solved a similar problem in generalizing Gödel's incompleteness theorem to non Σ_1–sound theories (Theorem 2.2). A similar construction will suffice for our present problem.

Theorem 1. Let X be any r.e. set. There is then a Σ_1 (Π_1) formula $\xi(x)$ which numerates X in T.

Proof. There is a primitive recursive relation R(k,m) such that X = {k: $\exists$mR(k,m)}. Let $\rho(x,y)$ be a PR binumeration of R(k,m). Let $\xi(x)$ be such that

(1) $\quad Q\vdash \xi(k) \leftrightarrow \exists y(\rho(k,y) \wedge \forall z{\le}y\neg Prf_T(\xi(k),z))$.

Then $\xi(x)$ is Σ_1. We are going to show that $\xi(x)$ numerates X in T.

Suppose first $k\in X$. There is then an m such that

(2) $\quad Q\vdash \rho(k,m)$.

Now, for *reductio ad absurdum*, suppose $T\nvdash \xi(k)$. Then $Q\vdash \neg Prf_T(\xi(k),p)$ for every p. It follows that

(3) $\quad Q\vdash \forall z{\le}m\neg Prf_T(\xi(k),z)$.

Combining (2) and (3) we get

$$Q\vdash \exists y(\rho(k,y) \wedge \forall z{\le}y\neg Prf_T(\xi(k),z)).$$

But then, by (1), $Q\vdash \xi(k)$ and so $T\vdash \xi(k)$ and we have reached the desired contradiction. Thus, $T\vdash \xi(k)$.

Next suppose $T\vdash \xi(k)$. Let p be a proof of $\xi(k)$ in T. Then $Q\vdash Prf_T(\xi(k),p)$ and so

(4) $\quad Q\vdash \forall z{\le}y\neg Prf_T(\xi(k),z) \to y < p$.

Suppose $k\notin X$. Then $Q\vdash \neg\rho(k,m)$ for every m. It follows that

(5) $\quad Q\vdash \neg\exists y{<}p\rho(k,y)$.

Combining (4) and (5) we get

$$Q\vdash \neg\exists y(\rho(k,y) \wedge \forall z{\le}y\neg Prf_T(\xi(k),z)),$$

whence, by (1), $Q\vdash \neg\xi(k)$ and so $T\vdash \neg\xi(k)$, impossible. Thus, $k\in X$ and we have shown that $\xi(x)$ numerates X in T.