Accessible Segments of the Fast Growing Hierarchy *

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Abstract. We examine two ways of "bootstrapping" segments of the fast growing hierarchy by autonomous generation. One method closes off at ε_0 with the provably recursive functionals of arithmetic, whereas the other exhausts the provably recursive functions of $\Pi_1^1 - CA_0$.

1   Introduction.

Autonomously generated hierarchies are constructed according to the principle: *proceed to level α if α is already coded or recognized at some earlier level.* The question addressed here is : how might this principle be applied in a sub-recursive (rate of growth) context, and what is the effect? Clearly it will depend upon the way in which we choose to subrecursively code or recognize countable ordinals. Typically this could be done by means of some "natural" scale or hierarchy $\alpha \mapsto f_\alpha$ such that the growth-rate of function f_α reflects its rank α. However this is a delicate matter since (in contrast with generalized recursion theory, where ordinal comparison is a fundamental property) one cannot expect to compare and compute ordinals by sub-recursive functions without reference to some prior given notation system. Either one simply accepts this situation and does the best one can (and we will, later, in section 4); or alternatively one could shift the goalposts and try to reconsider the problem in a more amenable, generalized setting. One suitable place to look is the theory of type two recursive functionals where, as noted by Kleene (1958) and in stark contrast with the recursive functions, a good notation-free hierarchy already exists - by classifying total recursive functionals according to the ordinal heights of their trees of unsecured sequences (see also Wainer (1995)). This opens the possibility of coding and comparing recursive ordinals in terms of the majorization relationship between certain descent-recursive functionals which represent them.

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* This paper presents newer material not surveyed in the author's conference lecture (much of which can already be found in ref.10), but expanding the same overall theme.