Convergence Laws for Random Graphs

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Abstract. A convergence law in logic pertains to a language and class of finite structures on which a probability measure has been assigned to each set of structures of a given size. It states that, for every sentence in the language, the probability that it holds for a random structure approaches a limit as the size of the structure grows. A 0-1 law states that this limit must be 0 or 1. The original convergence law was a 0-1 law for first-order logic and relational structures with a uniform probability distribution. This expository article shows how it has been extended by numerous authors to more powerful logics and the class of random structures known as random graphs. In many cases, the 0-1 law no longer holds, but a convergence law can still be proven. Full proofs are not given, but a uniform framework is provided which emphasizes ideas common to all the proofs.

1 Background

Let us begin by introducing the conventions we will use. $\mathcal{L}$ will denote a logic of some type τ. $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ will stand for a sequence of sets of τ-structures where all structures in $\mathcal{C}_n$ have universe $\{0, 1, \ldots, n-1\}$, which we will abbreviate as n. For every $n \in \omega$, pr_n is a probability measure on $\mathcal{C}_n$. For a sentence $\sigma \in \mathcal{L}$, we put $\mathrm{pr}(\sigma, n)$ for

$$\mathrm{pr}_n(\{\, \mathfrak{A} \models \sigma : \mathfrak{A} \in \mathcal{C}_n \,\}) \ .$$

In this article, we will examine the behavior of $\mathrm{pr}(\sigma, n)$ for growing n (given $\mathcal{L}$, $(\mathcal{C}_n)_{n\in\omega}$, and $(\mathrm{pr}_n)_{n\in\omega}$). The first theorems on this topic considered a first-order logic $\mathcal{L}$ of a purely relational type τ, where each $\mathcal{C}_n$ is the set of all τ-structures on n, and pr_n is the uniform distribution. For example, if $\mathcal{L}$ has a single relational symbol of arity r, then

$$\mathrm{pr}(\sigma, n) = \frac{|\{\, \mathfrak{A} \models \sigma : \mathfrak{A} \in \mathcal{C}_n \,\}|}{2^{n^r}} \ .$$

Theorem 1 Fagin [11], Glebskiĭ et al. [13]. *For every* $\sigma \in \mathcal{L}$,

$$\lim_{n\to\infty} \mathrm{pr}(\sigma, n) = 0 \ \mathit{or}\ 1 \ .$$

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