Noninterpretability of Infinite Linear Orders

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Abstract. We prove that no infinite linear order can be interpreted without parameters in any structure of the form $B \times B$, and describe applications of this result in recursion theory.

1 Squares

We are interested structures **A** which *fail* to satisfy the following property.

(ILOP) It is possible to interpret an infinite linear order in **A** without parameters.

The abbreviation stands for Infinite Linear Order Property. Interpretations of a structure **C** in a structure **A** are studied in [Ho 93]. In brief, an interpretation is a way to encode **C** into **A** using a finite collection of first–order formulas in the language of **A** as a decoding key. Elements of **C** are represented by tuples of a fixed length m of elements of B, modulo some definable equivalence relation. For the special case of an interpretation of a reflexive linear order Q in **A**, it is enough to consider a decoding key consisting of a single formula $\varphi(\overline{x}, \overline{y})$, also written as $\overline{x} \leq_Q \overline{y}$. Here $\overline{x}, \overline{y}$ are tuples of variables of length m, and, in **A**, the formula defines a linear pre–ordering on the domain $\{\overline{x} : \overline{x} \leq_Q \overline{x}\}$. For instance, $(\mathbb{Q}, \leq)$ can be interpreted in the ring $\mathbb{Z}$ using the formula

$$(z, w) \leq_Q (z', w') \equiv \psi_{\leq}(1, w) \wedge \psi_{\leq}(1, w') \wedge \psi_{\leq}(z \cdot w', z' \cdot w),$$

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