Model Theory of Modules

(Extended Abstract)

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This note is meant to give a brief account of some themes in the model theory of modules as they have developed since the subject's beginning. We shall survey the situation from the perspective that has grown in recent years out of an effort to formulate the theory in a way that stresses the common ground shared by the model theory of modules with other areas of mathematics, most notably representation theory and abelian category theory. We will take this opportunity to state the classical theorems of the theory in a context more general and hopefully more accessible to those working in other areas. On the other hand, this exposition hopes to illustrate how the theory of left exact functors, developed by Gabriel [4], figures into model theory. The two standard references for this subject are [7] and [12].

The seminal results of what has turned out to be the model theory of modules were attained by Wanda Szmielew [10] in her work on abelian groups, that is, modules over the ring of integers. Among other things, Szmielew characterized the complete theories of abelian groups. Quite sometime later, Eklof and Fisher [3] exploited the theory of algebraically compact (= pure-injective) abelian groups that had developed in the meantime to present Szmielew's classification in a more conceptual framework. The following result is not the classification itself but a result that generalizes nicely.

Theorem. Every abelian goup is elementarily equivalent to a direct product of pure-injective indecomposable abelian groups.

The pure-injective indecomposable abelian groups were classified by Kaplansky [5] into four groups:

Torsion free, divisible. The group of rational numbers Q.
Torsion, divisible. The Prüfer groups $Z(p^{\infty})$, p prime.
Torsion free, not divisible. The p-adic completions $\overline{Z_p}$ of the integers, p prime.
Torsion, not divisible. The cyclic groups of order a prime power $Z(p^n)$.

The theorem thus gives a complete list of abelian groups up to elementary equivalence. This list is however not without repetition so the problem that emerges is to describe how the pure-injective indecomposables relate to each other. More precisely, one must consider the following question.