Types and Indiscernibles in Finite Models

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Abstract. We consider L^k — first order logic restricted to k variables, and interpreted in finite structures. The study of classes of finite structures axiomatisable with finitely many variables has assumed importance through connections with computational complexity. In particular, we investigate the relationship between the size of a finite structure and the number of distinct types it realizes, with respect to L^k. Some open questions, formulated as finitary Löwenheim-Skolem properties, are presented regarding this relationship. This is also investigated through finitary versions of an Ehrenfeucht-Mostowski property.

1 Introduction

In this paper, we are concerned only with finite structures. That is, our logical formulas are interpreted in relational structures with finite domain. Interest in finite model theory has largely grown from the fact that there is a close connection between definability of classes of finite structures and their computational complexity. For instance, Fagin [10] showed that a class of finite structures is definable in existential second order logic if, and only if, it is decidable by a nondeterministic Turing machine in polynomial time (i.e. in the class NP). Similarly, many naturally arising complexity classes have been characterised as definability classes in appropriate logics.

These results raised the hope that model theoretic methods could be deployed to attack some of the notoriously open problems in complexity theory. Unfortunately, most methods and results developed in model theory fail to work when only finite structures are considered (see [12]). For instance, the compactness theorem for first order logic trivially fails, as do most of its consequences. Indeed, as we make the transition from arbitrary structures to finite structures, first order logic loses its central role. Two significant reasons can be discerned for this: on the one hand, first order logic is too strong; and on the other hand, first order logic is too weak.

First order logic is too strong in the sense that for every finite structure $\mathfrak{A}$, there is a first order sentence $\varphi_{\mathfrak{A}}$ that describes it up to isomorphism: that is, for any structure $\mathfrak{B}$, if $\mathfrak{B} \models \varphi_{\mathfrak{A}}$, then $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$. This means that the relation $\equiv$ of elementary equivalence is trivial on finite structures — it coincides with the isomorphism relation. Since a large part of model theory can arguably be described as the study of the structure of the elementary equivalence relation