The Number of Path-Components of a Compact Subset of $\mathbb{R}^n$

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§0. Introduction

This paper is concerned with the following question. Assume $\neg CH$; does there exist a compact set $K \subset \mathbb{R}^n$ such that K has exactly $\aleph_1$ path-components? For $\mathbb{R}^3$, the answer is yes. For $\mathbb{R}^2$, the answer is no, assuming a weak large cardinal axiom (which may or may not be necessary).

The proof of both results is descriptive set theoretic. Indeed, the motivation for asking the question is descriptive set theoretic. The same question for components, rather than path-components, would be a silly question; it is obvious (at least to descriptive set theorists) that the answer is no. It is also obvious that it is not possible that $2^{\aleph_0} \geq \aleph_3$ and that there is a compact $K \subset \mathbb{R}^n$ with $\aleph_2$ path-components. But the question as posed above does not seem to be a silly question. One of the purposes of this paper is to present the descriptive set theoretic point of view, and hopefully convince the reader that these "obvious" facts really are obvious. Two references for descriptive set theory are Kechris [13] and Moschovakis [17], and we follow their notation and terminology.

In both the $\mathbb{R}^3$ and $\mathbb{R}^2$ cases, we have results that are stronger than those stated above. In both cases, the size of the continuum is irrelevant and the theorem – properly stated – is nontrivial even if CH is true. These theorems will be given in §2. For $\mathbb{R}^3$, there is a more general theorem, a precise version of the following: Any $\boldsymbol{\Sigma}^1_1$ equivalence relation can be coded up as the equivalence relation of being in the same path-component of K, for some compact $K \subset \mathbb{R}^3$. From this it easily follows that there is a $K \subset \mathbb{R}^3$ with $\aleph_1$ path-components. That general theorem has other applications as well, one of which answers a question of Kunen-Starbird [14]. This paper is largely an explanation of the statement of these stronger theorems, and of the larger mathematical theory of which they are a part, that is, the descriptive set theory of equivalence relations. In the $\mathbb{R}^3$ case we say virtually nothing about the proof. In the $\mathbb{R}^2$ case we give an outline of the proof (§§6,7), containing several gaps, and using a stronger large cardinal axiom than required.

The author plans to some day write a long paper about path-connectedness, simple connectedness and descriptive set theory (Becker [3]). The results announced here will appear there with complete proofs. Most of Becker [3] will be concerned with calculating the complexity, with respect to the projective

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