Rather Classless, Highly Saturated Models of Peano Arithmetic

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Every saturated model of Peano Arithmetic having cardinality λ has 2^λ classes. Therefore, no saturated model of PA is rather classless. In other words, if $\kappa = \lambda$, then there are no rather classless, λ-saturated models of PA having cardinality κ. However, as long as λ is regular and $\kappa > \lambda$, there are no obstacles to the existence of rather classless, λ-saturated models of PA of cardinality κ other then there being no λ-saturated models of PA of cardinality κ at all. This is the content of the following theorem, which is the main result of this paper.

Theorem *If λ is regular, $\mathcal{N} \models$ PA is λ-saturated and $\lambda < |N|$, then there is a rather classless, λ-saturated $\mathcal{M} \succ \mathcal{N}$ such that $|M| = |N|$.*

The first rather classless, highly saturated models of Peano Arithmetic were constructed by Keisler [5]. His general theorem, specialized to models of PA, yields that whenever T is a consistent completion of PA, $\lambda^{<\lambda} = \lambda \geq \aleph_1$, and the combinatorial principle $\diamondsuit_{\lambda^+}$ holds, then there are rather classless, λ-saturated models of T of cardinality λ^+ (which, moreover, are λ^+-like). More rather classless, highly saturated models of PA can be obtained from a general theorem of Shelah (Theorem 12 of [8]) which, when specialized to models of PA, yields the following: If T is a consistent completion of PA, κ is the successor of a regular cardinal, and λ is a regular cardinal such that $\aleph_1 \leq \lambda < \kappa = \kappa^{<\lambda}$, then T has a rather classless, λ-saturated model of cardinality κ.

Kaufmann [3], assuming the combinatorial principle $\diamondsuit$, proved that there are $\aleph_1$-like, rather classless, recursively saturated models of each consistent completion of PA. Subsequently, this dependence on $\diamondsuit$ was eliminated by Shelah [8].

Rather classless, recursively saturated models of PA of each uncountable cardinality were constructed in Schmerl [7]. These models could be made

[1]Received september 26, 1996; revised version september 22, 1997.