Arithmetizing proofs in analysis

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1 Introduction

In this paper we continue our investigations started in [15] and [16] on the question:

What is the impact on the growth of extractable uniform bounds the use of various analytical principles Γ in a given proof of an $\forall\exists$–sentence might have?

To be more specific, we are interested in analyzing proofs of sentences having the form

$$(1)\ \forall u^1, k^0 \forall v \leq_\rho tuk \exists w^0 A_0(u,k,v,w),$$

where A_0 is a quantifier–free formula[2] (containing only u, k, v, w as free variables) in the language of a suitable subsystem $\mathcal{T}^\omega$ of arithmetic in all finite types, t is a closed term and $\leq_\rho$ is defined pointwise (ρ being an arbitrary finite type).

From a proof of (1) carried out in $\mathcal{T}^\omega$ one can extract an effective uniform bound Φuk on $\exists w$, i.e.

$$(2)\ \forall u^1, k^0 \forall v \leq_\rho tuk \exists w \leq_0 \Phi uk\, A_0(u,k,v,w),$$

where the complexity (and in particular the growth) of Φ is limited by the complexity of the system $\mathcal{T}^\omega$ (see [13],[15]).

By the predicate 'uniform' we refer to the fact that the bound Φ does not depend on $v \leq_\rho tuk$.

In [13] we have discussed in detail, how sentences (1) arise naturally in analysis and why such uniform bounds are of numerical interest (e.g. in the context of approximation theory).

[1]Received September 96; revised version February 97.

[2]Throughout this paper $A_0, B_0, C_0, \ldots$ always denote quantifier–free formulas.