IP Current Developments in Mathematics, 2004

A General Fredholm Theory and Applications

H. Hofer

The theory described here results from an attempt to find a general abstract framework in which various theories, like Gromov-Witten Theory (GW), Floer Theory (FT), Contact Homology (CH) and more generally Symplectic Field Theory (SFT) can be understood from a general point of view. Let us describe the general landscape in a somewhat oversimplified form. The common feature (with the exception of GW which has less structure) is the fact that we have infinitely many different Fredholm problems defined on spaces with boundary with corners, where the boundary strata can be explained in terms of products (or more generally fibered products) of other problems (on the list). In oversimplified form, the solution sets are zeros of a section f of some bundle $\tau : Y \to X$, where the space has a boundary ∂X, and where moreover there exists a recipe (or even many recipes) to construct from two given solutions[1] x' and x'' of $f = 0$ a new solution, say the product, $x = x' \circ x''$. The recipes for constructing new solutions are defined even for non-solutions and ∂X is precisely the space of points which are products. Hence we have

$$\partial X = X \circ X.$$

Moreover, if we denote the restriction of f to ∂X by ∂f and define $f \circ f$ on ∂X as the set-valued section

$$f \circ f(x) = \{f(x') \circ f(x'') \mid x = x' \circ x''\},$$

then we say that f is compatible with the recipe $\circ$ provided

$$\partial f = f \circ f.$$

Assuming f to be compatible the $\circ$-structure generates a certain amount of algebra which can be captured on a rather rudimentary level. Then the more sophisticated algebra we see in the description of SFT can be

Research partially supported by NSF grants DMS-0102298 and DMS-0505968.

[1] In general one requires them to satisfy a compatibility condition.