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Some recent results on representations of p-adic special orthogonal groups

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1. Motivation and results

1.1. Global motivation. Let V be a vector space over $\mathbb{Q}$, of finite dimension d, and let q be a non-degenerate quadratic form on V. We suppose given an orthogonal decomposition $V = D \oplus W$, where D is a line. Denote by G, resp. H, the special orthogonal group of V, resp. W. The group H is a subgroup of G, it is the subgroup of elements which act by identity on D. We assume, for instance, d even, and, to simplify, $d \geq 4$.

Denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$. The group $G(\mathbb{A})$ acts by right translations on the space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of square-integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. Consider a closed subspace $\mathcal{V} \subset L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, invariant by $G(\mathbb{A})$, such that the representation π of $G(\mathbb{A})$ on $\mathcal{V}$ is irreducible (there is no closed invariant subspace distinct from $\mathcal{V}$ and $\{0\}$). Such representation π is called automorphic of the discrete spectrum. The quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ is not compact in general, but we can define the notion of "rapid decay" for a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$. Assume that the subset $\mathcal{V}^0$ of elements of rapid decay of $\mathcal{V}$ is dense in $\mathcal{V}$. Then we say that π is cuspidal. Let π be such automorphic cuspidal irreducible representation of $G(\mathbb{A})$ on a subspace $\mathcal{V} \subset L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Similarly, let σ be an automorphic cuspidal irreducible representation of $H(\mathbb{A})$ on a subspace $\mathcal{W} \subset L^2(H(\mathbb{Q})\backslash H(\mathbb{A}))$. For $\varphi \in \mathcal{V}^0$ and $\psi \in \mathcal{W}^0$, we define the integral

$$J(\psi, \varphi) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \overline{\psi(h)}\varphi(h)\, dh.$$

It is absolutely convergent and define a sesquilinear form J on $\mathcal{W}^0 \times \mathcal{V}^0$. A conjecture of Gross and Prasad ([GP], conjecture 14.8) says (in particular) that, if this sesquilinear form is non-zero, then a special value of some L-function is non-zero. What is this L-function? Of course, the notion of automorphic representation of the discrete spectrum is defined for all reductive groups over $\mathbb{Q}$ (some minor modifications are needed when the