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Phase Transitions, Minimal Surfaces and A Conjecture of De Giorgi[1]

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1. Introduction

A central problem in the area of PDEs is the study of global solutions, that is solutions defined in the whole space. This problem arises naturally for example when one studies the possible types of behaviors a solution might have at a given point. Focusing near such a point by dilating the picture more and more, we end up with a PDE in the whole space. In other cases we might have a differential equation with oscillating terms on a small scale depending on a parameter ε. In order to understand the behavior at unit scale as the parameter $\varepsilon \to 0$ we often need to dilate the picture and solve some PDE in the whole space. As a consequence, the study of global solutions is crucial when dealing with singularities, local behavior and homogenization limits of PDEs.

In this paper we present symmetry results for global solutions to certain semilinear (or fully nonlinear) elliptic equations of the type

$$\triangle u = f(u)$$

which arise in the theory of phase transitions.

Let us briefly introduce the typical physical model for a phase transition. Imagine in a domain Ω we have a two-phase fluid whose density at a point x we denote by $\rho(x)$. Assume its energy density is given by a double-well potential $W(\rho(x))$ say with minima at ρ_1, ρ_2 i.e.

$$W(\rho_1) = W(\rho_2) = 0, \quad W(s) > 0 \quad \text{if } s \neq \rho_1, \rho_2.$$

The densities ρ_1 and ρ_2 correspond to the stable fluid phases. Then the energy of the fluid is given by the integral

$$\int_\Omega W(\rho(x))\, dx.$$

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