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# Recent Progress in GW-Invariants of Calabi-Yau Threefolds

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The last two decades is a period when exchange of ideas between algebraic geometry and theoretical physics has created several fast advancing branches in algebraic geometry. The research on Gromov-Witten (GW-) invariants is a notable example of such branches.

GW-invariants is a mathematical foundation to the type II Super-String theory. In terms of algebraic geometry, it is a "virtual" counting of holomorphic maps from algebraic curves to a projective variety. Via a multiple cover relation, the GW-invariants will provide a "virtual" counting of holomorphic embedded curves.

Properties of GW-invariants differ according to the kinds of the varieties. One indicator is the Kodaira dimensions. The Theory of GW-invariants of Fano varieties is the complete opposite of those with positive Kodaira dimensions, the former is much richer than the later. However, the GW-invariants of Calabi-Yau varieties are the most mysterious and challenging. In this note, we will comment on the recent progress in research on the GW-invariants of Calabi-Yau threefolds. Due to the interest of the author, we will only report on progress via algebraic geometry approach.

The investigation of GW-invariants went through several periods of development. Inspired by the $\sigma$-model in Super-String theory [**37**] and using pseudo-holomorphic maps introduced by Gromov, Ruan-Tian [**32**] constructed the GW-invariants of semi-positive, including Calabi-Yau, symplectic manifolds. Later, Li-Tian [**23**] and Behrend-Fantechi [**5, 4**] constructed the GW-invariants of all smooth projective varieties via algebraic geometry. Their construction is based on the construction of virtual cycles of the moduli of Kontsevich's stable morphisms, and the GW-invariants are the degree of these cycles paired with tautological classes of the moduli space. The symplectic construction of GW-invariants was completed by the work of Fukaya-Ono [**9**] and Li-Tian [**24**], Ruan [**31**] and Siebet [**33**]; also a much detailed technical account by Zinger [**42**]. The equivalence