

# The Fundamental Lemma: What is it and what do we know?

Stephen DeBacker

## 1. Introduction

The Fundamental Lemma (FL) was first proposed about twenty years ago by Langlands and Shelstad [29] as part of the trace-theoretic approach to the Langlands program. The FL is a rather delicate statement, which, to judge by its name, was thought to be both important and not particularly deep. The technical nature of the FL makes it rather difficult to understand and appreciate. Thus, rather than give a technical presentation full of correct definitions, we will instead attempt in this discussion to convey the flavor of the subject. As such, lies will occur<sup>1</sup>; the interested reader may consult the original papers for complete details.

Our discussion is divided into two parts. In the first part, we shall try to motivate the statement of the FL by looking at the (conjectural) Local Langlands Correspondence (LLC).

There are at least two interesting classes of distributions on a  $p$ -adic group: orbital integrals and characters. Although the FL is concerned with the former, most people are more comfortable with the latter. Thus, in the first part of our discussion, we will use characters and the framework of the (conjectural) LLC to introduce most of the formalism required to state the FL. The LLC, the existence of which has been proven for general linear groups by Harris–Taylor and Henniart, is a finite-to-one correspondence from the set of (equivalence classes of) irreducible smooth representations of a  $p$ -adic group to the set of (equivalence classes of) maps from the Weil–Deligne group into the Langlands dual group of the  $p$ -adic group. Verifying the existence of the LLC is the fundamental open problem in local harmonic analysis for  $p$ -adic

---

Supported by National Science Foundation Grant Number 0345121.

<sup>1</sup>I hope only intentionally.