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Adelic dynamics and arithmetic quantum unique ergodicity

Elon Lindenstrauss

1. Introduction

Let M be a complete Riemannian manifold with finite volume which we initially assume to be compact. Then since M is compact, $L^2(M)$ is spanned by the eigenfunctions of the Laplacian Δ on M.

Many interesting questions can be asked about these eigenfunctions and their properties, and of these we focus on one, quantum ergodicity, which to the best of my knowledge was first considered by A.I. Šnirel′man, and was substantially sharpened in the work of Z. Rudnick and P. Sarnak which deals with the equidistribution properties of these eigenfunctions.

Specifically, let ϕ_n be a complete orthonormal sequence of eigenfunctions of Δ ordered by eigenvalue. These can be interpreted for example as the steady states for Schroedinger's equation

$$i\frac{\partial\psi}{\partial t} = \Delta\psi$$

describing the quantum mechanical motion of a free (spinless) particle on M. According to Bohr's interpretation of quantum mechanics $\tilde{\mu}_n(A) := \int_A |\phi_n(x)|^2 \, d\mathrm{vol}(x)$ is the probability of finding a particle in the state ϕ_n inside the set A at any given time.

A.I. Šnirel′man, Y. Colin de Verdière and S. Zelditch [**26, 8, 30**] have shown that whenever the geodesic flow on M is ergodic, for example if M has negative curvature, there is a subsequence n_k of density one on which $\tilde{\mu}_{n_k}$ converge in the weak* topology to the uniform measure $\frac{1}{\mathrm{vol}(M)} d\mathrm{vol}_M$[1]. This phenomenon is called quantum ergodicity.

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[1]I.e., for every continuous (compactly supported in more general situations) function f one has that $\int f d\mu_{n_k} \to \frac{1}{\mathrm{vol}(M)} \int f d\mathrm{vol}_M$.