A THEOREM ON FUNCTIONS OF CHARACTERISTIC FUNCTIONS AND ITS APPLICATION TO SOME RENEWAL THEORETIC RANDOM WALK PROBLEMS

WALTER L. SMITH
UNIVERSITY OF NORTH CAROLINA

Summary

This paper is concerned with three main problems and their interrelationship. A function $M(x)$ belongs to the "moment-class" $\mathfrak{M}^*$ if it is nonnegative and nondecreasing on $[0, \infty)$, if $M(x + y) \leq M(x)M(y)$ for all $x, y \geq 0$, and if $M(2x) = O(M(x))$ for all $x \geq 0$. The class $\mathfrak{B}^\ddagger(M; \nu)$, for any real $\nu \geq 0$, is the Banach algebra of functions which are Fourier-Stieltjes transforms of functions $B(x)$ of bounded total variation such that $\int_{-\infty}^{+\infty} |x|^\nu M(|x|)|dB(x)| < \infty$. Our first main result, theorem 3, demonstrates that if a characteristic function belongs to $\mathfrak{B}^\ddagger(M; \nu)$, then it has a Taylor expansion whose remainder term may involve a nonintegral power of $|\theta|$ and a member of some subalgebra $\mathfrak{B}^\ddagger$ whose "parameters" depend on a variety of details which we suppress in this summary. A version of the Wiener-Pitt-Lévy theorem on analytic functions of functions of $\mathfrak{B}^\ddagger(M; \nu)$ is then given, and from this and our results about Taylor expansions of characteristic functions, we obtain our second main result, a "Master Theorem" (theorem 1). This Master Theorem considers a certain rational form involving several characteristic functions and shows that under appropriate conditions it will be in some algebra $\mathfrak{B}^\ddagger(M; \nu)$; the form has been chosen as being liable to arise in various investigations in the theory of random walks.

The Master Theorem and the results about characteristic functions are then applied to a general problem of a renewal-theoretic nature. Suppose $\{X_n\}$ is an infinite sequence of independent and identically distributed random variables such that $0 < \mathcal{E} X_n < \infty$. Let the characteristic function of X_n belong to $\mathfrak{B}^\ddagger(M; \nu)$ for some $M \in \mathfrak{M}^*$ and some $\nu \geq 0$. Then what can be said about the asymptotic nature of, for instance,

$$S_\ell(x) \equiv \sum_{n=0}^{\infty} \binom{-(\ell - 1)}{n} P\{X_1 + X_2 + \cdots + X_n \leq x\},$$

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