CROSS-SECTIONS OF ORBITS AND THEIR APPLICATION TO DENSITIES OF MAXIMAL INVARIANTS

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1. Introduction and summary

Let G be a group of one-to-one transformations of the sample space X onto itself. A maximal invariant is a function constant on orbits and distinguishing orbits. If G leaves a certain statistical problem invariant and an invariant procedure is to be selected, it is necessary first to solve the problem of how to obtain the distribution of a maximal invariant, given any distribution on X. One of the possible methods consists of writing this distribution as an integral over the group G. This method has been promoted notably by Stein [10], [11], Karlin [6], and James [5], but does not seem to have been used very much in the literature (among the exceptions, see [3], [7]) in spite of the fact that the method has several advantages. Unfortunately, although some specific problems have thus been treated, there does not seem to exist much in the form of a general theory. This paper is intended as a step in that direction. Some new theorems will be presented and several examples given.

The principal tool used in this paper that makes things work is the so-called *cross-section* of orbits, local or global (precise definitions of various terms will be given in section 2). A global cross-section is a subset Z of X such that every orbit intersects Z at exactly one point, in addition to a few other properties to be defined in section 2. A local cross-section at x is a global cross-section for an open, invariant neighborhood of the orbit passing through x. If a global cross-section Z exists, it is possible to convert an integral $\int_X p\, d\mu$ (μ is Lebesgue measure) into an iterated integral of the form $\int_Z \nu_Z(dz) \int_G p(gz)\nu_G(dg)$, where ν_Z and ν_G are certain measures on Z, G, respectively. For any global cross-section Z there is a natural maximal invariant, namely the function that associates to every orbit its intersection with Z. For any distribution P on X, with density p with respect to Lebesgue measure, the distribution of the maximal invariant is then a distribution on Z given by $\nu_Z(dz) \int p(gz)\nu_G(dg)$. The exact nature of the measures ν_Z and ν_G will be given in sections 4 and 5.

In many statistical problems the primary interest is in the probability ratio of a maximal invariant, given any two densities p_1 and p_2. It is then not necessary

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