A NOTE ON MAXIMAL POINTS OF CONVEX SETS IN ℓ_∞

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1. Introduction

The problem of characterizing maximal points of convex sets often arises in the study of admissible statistical decision procedures, of efficient allocation of economic resources (cf. Koopmans, [4], chapter 1, and references given there), and of mathematical programming (cf. Arrow, Hurwicz, and Uzawa, [2]).

Let C be a convex set in a finite dimensional vector space, partially ordered coordinate-wise (that is, for $x = (x_i)$ and $z = (z_i)$, $x \geq z$ means that $x_i \geq z_i$ for every coordinate i). Let D be the set of all strictly positive vectors (namely vectors all of whose coordinates are strictly positive); further, let B be the set of vectors in C that maximize $\sum_i y_i x_i$ for some vector $y = (y_i)$ in D. It is obvious that every vector in B is maximal in C with respect to the partial ordering $\leq$. One can also show that every vector that is maximal in C also maximizes $\sum_i y_i x_i$ on C for some nonnegative vector y. Arrow, Barankin, and Blackwell [1] showed further that every vector maximal in C is in the (topological) closure of B. They also gave an example (in 3 dimensions) in which a vector in the closure of B (and in C) is not maximal in C.

The purpose of this note is to generalize the Arrow-Barankin-Blackwell result to the case of ℓ_∞, the space of bounded sequences topologized by the sup norm. In this generalization, however, the set C is assumed to be compact.

2. The theorem

Let X denote ℓ_∞, that is, the Banach space of all bounded sequences of real numbers, with the sup norm topology, where the norm of $x = (x_i)$ in X is

$$\|x\| \equiv \sup_i |x_i|. \tag{2.1}$$

For x in X, I shall say that $x \geq 0$ if $x_i \geq 0$ for every i, and that $x > 0$ if $x \geq 0$ but $x \neq 0$. Also, for $x^1 = (x_i^1)$ and $x^2 = (x_i^2)$ in X, I shall say that $x^1 \geq x^2$ if $x^1 - x^2 \geq 0$ (and so on for $x^1 > x^2$).

A point $\hat{x}$ in a subset C of X will be called *maximal in* C if there is no x in C for which $x > \hat{x}$.

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