THE EXISTENCE OF STATIONARY MEASURES FOR CERTAIN MARKOV PROCESSES

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1. Introduction

We consider a Markov process x_n, $n = 0, 1, \cdots$. The random variables x_n belong to an abstract set X in which a Borel field B is defined, X itself being an element of B. It is assumed throughout this paper that B is separable; that is, B is the Borel extension of a denumerable family of sets. The transition law of the process is given by a function $P(x, E) = P^1(x, E)$, this function being interpreted as the conditional probability that $x_{n+1} \in E$, given $x_n = x$. The n-step transition probability is designated by $P^n(x, E)$. When conditional probabilities are used below, it will usually be understood that they are the ones uniquely determined by the transition probabilities. The sets in B will sometimes be called "measurable sets."

Throughout this paper a "measure" will mean a countably additive set function, defined on the measurable sets, nonnegative, and not identically 0. (The words "countably additive" will sometimes be repeated for emphasis.) A "probability measure" or "probability distribution" will be a measure of total mass 1. Notice that we do not require measure to be finite. A "sigma-finite" measure is a measure such that X is the union of a denumerable number of sets, each of which has a finite measure.

Various conditions are known which imply the existence of a probability measure $Q(E)$ which is a stationary distribution for the x_n-process; that is, Q satisfies, for each measurable E,

$$Q(E) = \int_X Q(dx) P(x, E) . \tag{1.1}$$

If x_0 has this distribution, so has x_n for every n. Two sets of conditions for the existence of such a probability measure were given by Doeblin. One set is discussed in Doob (see pp. 190 ff. in [7]). A more general set is given in [6].

There are many situations where there is no probability measure satisfying (1.1), but where a solution can be found if $Q(X) = \infty$ is allowed. The simplest example is the random walk where x_n takes integer values, and can increase or decrease by 1, with probabilities 1/2 each, at each step. In this case a solution to (1.1) can be obtained by assigning to any set of integers a Q-measure equal to the number of integers in the set. All integers are "equally probable."

In this paper a solution of (1.1) will always mean a sigma-finite measure Q which satisfies (1.1) for every measurable set E. The principal result, contained in theorem 1, is

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