BAYES AND MINIMAX ESTIMATES FOR QUADRATIC LOSS FUNCTIONS

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1. Introduction

This paper is concerned with certain technical aspects of Wald's theory of minimax risk and cursory familiarity with that theory, such as may be had by reading the introductory sections of [1] and [16], is accordingly assumed.

In the present stage of this theory it seems appropriate to invest some activity in the exploration of minimax problems of an intermediate level of generality. We in particular thought it might be fruitful to explore the problem of estimating a real parameter θ where the loss as a function of θ and the estimate g is of the special form

$$W(g, \theta) = \lambda(\theta)(g - \theta)^2 ; \quad \lambda(\theta) > 0 . \tag{1.1}$$

Restriction to a single parameter is of course a convenience of the moment, for it would be surprising if any results obtained here could not be extended directly to quadratic forms. Interest in (1.1) is aroused by the power series consideration that any smooth nonnegative risk function can be expressed approximately in that form. We were further motivated to study (1.1) because a few very general results concerning it (given in section 2 below) led us to hope that we might discover a considerable body of theory at that level of generality. But thus far we have progressed beyond the results of section 2 only in two special contexts. The first of these special contexts (treated in section 3) is that in which θ is a parameter of translation and λ is constant. It is shown that such decision problems are closed, have minimax solutions with the symmetry which would be expected, and which are explicitly computable in terms of conditional expectations. There is every reason to believe that these results can be extended to many other loss functions for which the loss depends only on $g - \theta$.

The second special context in which we have studied (1.1) (treated in sections 4 and 5) pertains to families of distributions on the real line of the form $\beta(\tau) \exp [x\tau]\, d\psi(x)$, with $\theta = E(x \mid \tau)$, called exponential families. These exponential families are more versatile than would appear at first glance, as is explained early in section 4. If for one of these families $\lambda(\theta)$ is so chosen that $\lambda(\theta)V(x \mid \theta) = 1$ and if ψ is such that the range of τ can be the whole real line, then x is an admissible minimax estimate of θ if the range of τ is indeed taken to be the whole real line. If, further,

Part of the research contained in this paper was completed in the summer of 1949 while both authors were in the employ of the RAND Corporation, Santa Monica, California, and the remainder was completed under the sponsorship of Office of Naval Research projects at Stanford University and the University of Chicago.