AN EQUALITY IN STOCHASTIC PROCESSES

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1. Introduction

The equality presented in this paper is extremely simple, but has proven useful on a number of occasions in deriving certain transition probabilities where other approaches such as the Laplace transform or the generating function become untidy (see, for example, [7]). The equality is not completely unknown; it has appeared, in a slightly different form, in a two dimensional process ([2], p. 102); and it is obvious for the simple Poisson process. The purpose of this paper is to state it in a general form and to demonstrate its validity and usefulness with a number of examples.

2. The equality

Let $\{X(t);\, t \in T\}$ be a time dependent Markov process defined over the interval T: $[0, \infty)$. For each $t \in T$, the random variable $X(t)$ assumes nonnegative integer values with the transition probability

$$P_{ik}(t_0, t) = Pr\{X(t) = k | X(t_0) = i\}, \qquad 0 \leqq t_0 \leqq t < \infty,\ i \leqq k;\ i, k = 0, 1, \cdots . \tag{1}$$

Our discussion is related only to nondecreasing processes where the value of $X(t)$ is increased by the occurrence of an event (for example, the pure birth process), or the nonincreasing processes (for example, the pure death process). The equality will be presented only for the former cases. However, an example of the pure death process will be given in Section 3.

For each i, we assume the existence of a continuous function $\lambda_i(\tau)$ such that

$$\frac{\partial}{\partial t} P_{ij}(\tau, t)\bigg|_{t=\tau} = \begin{cases} \lambda_i(\tau) & \text{for } j = i + 1, \\ -\lambda_i(\tau) & \text{for } j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It follows that the transition probabilities in (1) satisfy the forward differential equations

$$\begin{aligned} \frac{d}{dt} P_{ii}(t_0, t) &= -\lambda_i(t) P_{ii}(t_0, t), \\ \frac{d}{dt} P_{ik}(t_0, t) &= -\lambda_k(t) P_{ik}(t_0, t) + \lambda_{k-1}(t) P_{i,k-1}(t_0, t), \\ &\qquad i \leqq k;\ i, k = 0, 1, \cdots . \end{aligned} \tag{3}$$