DIFFERENTIAL GAMES

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1. Introduction

In this paper, we shall give a brief account of the main ideas in the theory of differential games. Our presentation will be limited to two player, zero sum, differential games, except for a few words regarding the situations where there are more than two players or where the game is not zero sum.

To begin with, recall the game formulation when there are two players, *I* and *II*. The *actions* (*strategies*) available to *I* are represented by a set $A = \{\alpha\}$, whereas those available to *II* are described by the set $B = \{\beta\}$. There is specified a *payoff* function $P : A \times B \twoheadrightarrow R$, and *I* chooses α in A to maximize P while *II* chooses β in B to minimize P. In general, we know that the order in which the choices are made is essential, and we can only assert that

$$\sup_A \inf_B P(\alpha, \beta) \leqq \inf_B \sup_A P(\alpha, \beta). \tag{1.1}$$

When equality holds in (1.1), we say that the game (P, A, B) has a *saddle value*, and this common number is called the (saddle) value of the game. If there exist α^* in A and β^* in B such that,

$$P(\alpha, \beta^*) \leqq P(\alpha^*, \beta^*) \leqq P(\alpha^*, \beta), \qquad \alpha \in A, \beta \in B \tag{1.2}$$

then we say that (α^*, β^*) constitute a *saddle point* for the game. It is easy to see that in this case the game has a saddle value, and it is equal to $P(\alpha^*, \beta^*)$. Let us call games of this kind *matrix games*, since P has an obvious matrix representation when A and B are finite sets. The theory of matrix games started with the result of von Neumann, and since then many generalizations have appeared. A typical result is the following.

THEOREM 1.1. *Suppose A and B are convex, compact topological spaces. Suppose for fixed β in B, $P(\cdot, \beta)$ is concave, and upper semicontinuous, over A, and for fixed α in A, $P(\alpha, \cdot)$ is convex, and lower semicontinuous, over B. Then, the game (P, A, B) has a saddle point.*

Although the concave-convex assumption on P can be weakened slightly [14], it appears that in general this assumption (or an equivalent hypothesis) is essential [12]. This is in apparent striking contrast with the situation in differential games.

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