LOSE A DOLLAR OR DOUBLE YOUR FORTUNE

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1. Summary and introduction

A gambler with initial fortune x (a positive integer number of dollars) plays a sequence of identical games in which he loses one dollar with probability π, $0 < \pi < 1$, and doubles his fortune with probability $1 - \pi$. Playing continues until, if ever, the gambler is ruined (his fortune drops to zero). Let q_x denote the probability that the gambler starting with initial fortune x will eventually be ruined. Then, q_x satisfies the difference equation

$$q_x = \pi q_{x-1} + (1 - \pi) q_{2x}, \qquad x = 1, 2, \cdots, \tag{1.1}$$

with boundary conditions $q_0 = 1$ and $\lim_{x\to\infty} q_x = 0$. In Section 2, a solution to this difference equation subject to these boundary conditions is explicitly exhibited, and it is shown that there is only one such solution. In Section 3, the equation is extended to allow arbitrary noninteger values for the fortune, and again a solution is found. Section 4 contains several other extensions.

Equation (1.1) arises in connection with the following more general gambling problem described in [2]. A gambler is confronted with a sequence of games affording him even money bets at varying probabilities of success, $p_1, p_2, \cdots$, chosen independently from a distribution function F known to the gambler. The probability of winning the jth game p_j is told to the gambler after he plays game $j - 1$ and before he plays game j. The gambler must decide how much to bet in the jth game as a function of the past history, his present fortune, and the win probability p_j. He may bet any amount not exceeding his present fortune; however, he must bet at least one dollar on each game (called Model 2 in [2]). The problem of the gambler is to choose a betting system (a sequence of functions $b_1, b_2, \cdots$, where b_j is the amount bet in game j) that minimizes the probability of eventual ruin. Theorems relating to the dynamic programming solution of this problem may be found in Truelove [4].

Let q_x denote the infimum, over all betting systems, of the probability of ruin given the initial fortune x. It was shown in [2] that q_x tends to zero exponentially as x tends to infinity, and it was conjectured that, for some $0 < r < 1$ and $c > 0$,

$$q_x r^x \to c \qquad \text{as} \quad x \to \infty. \tag{1.2}$$

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