DIFFUSION PROCESSES

DANIEL W. STROOCK and S. R. S. VARADHAN
COURANT INSTITUTE, NEW YORK UNIVERSITY

1. Introduction

One of the major problems in the theory of diffusion processes is to construct the process for a given set of diffusion coefficients. A diffusion process in R^d is hopefully determined by the two sets of coefficients

$$(1.1)\qquad \begin{aligned} a &= a(t, x) = \{a_{ij}(t, x)\}, && 1 \leqq i, j \leqq d,\ t \in [0, \infty),\ x \in R^d, \\ b &= b(t, x) = \{b_j(t, x)\}, && 1 \leqq j \leqq d,\ t \in [0, \infty),\ x \in R^d. \end{aligned}$$

Here a is a positive semidefinite symmetric matrix for each t and x, and b is a d vector for each t and x. There are various ways of describing exactly what we mean by a diffusion process corresponding to the specified set of coefficients. We shall adopt the following approach.

Let Ω be the space of R^d valued continuous functions on $[0, \infty)$. The value of a function $\omega = x(\cdot)$ in Ω at time t will be denoted by $x(t)$. The σ-field generated by $x(s)$ for $t_1 \leqq s \leqq t_2$ will be denoted by $M_{t_2}^{t_1}$. If $t_1 = 0$, we will denote this by M_{t_2} and by M^{t_1} in case $t_2 = \infty$, where M is the σ-field generated by $x(s)$ for $0 \leqq s < \infty$. The space Ω can be viewed as a complete separable metric space, with uniform convergence on bounded intervals defining the topology. Then M is the Borel σ-field in Ω. A stochastic process with values in R^d, defined for $t \geqq t_0$, is a probability measure on (Ω, M^{t_0}).

Given the coefficients $\{a_{ij}(t, x)\}$ and $\{b_j(t, x)\}$, we define an operator L_t acting on functions $f(x) \in C_0^\infty(R^d)$ by

$$(1.2)\qquad (L_t f)(x) = \frac{1}{2} \sum a_{ij}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum b_j(t, x) \frac{\partial f}{\partial x_j}.$$

We say that a measure P is a solution to the Martingale problem corresponding to the given coefficients, starting at time t_0 from the point x_0 if

(a) P is a probability measure on (Ω, M^{t_0}) such that $P[x(t_0) = x_0] = 1$, and

(b) for each $f \in C_0^\infty(R^d)$, $f(x(t)) - \int_{t_0}^t (L_s f)(x(s))\, ds$ is a martingale relative to $(\Omega, M_t^{t_0}, P)$.

Under suitable conditions on the coefficients a and b, one should attempt to answer the following questions:

(1) For each t_0 and x_0, does a solution P_{t_0, x_0} exist?

Results obtained at the Courant Institute of Mathematical Sciences, New York University; this research was sponsored by the US Air Force Office of Scientific Research, Contract AF-49(638)-1719.